MULTIPLE SCATTERING OF ELECTROMAGNETIC WAVES BY RANDOM SCATTERERS OF FINITE SIZE

by

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Contents

Abstract

This report considers the propagation of electromagnetic waves in a random medium. When the randomness is caused by the presence of discrete, identical scattering objects embedded in a homogeneous medium, the problem is formulated in terms of multiply scattered fields. This type of formulation was first given in 1945 by Foldy, who introduced the concept of the configurational average. Since then much work has been done on the subject with valuable contributions from Lax, Twersky, Waterman and Truell. However, the treatments have been generally restricted to scalar waves and scatterers of small size. The present investigation extends the work to vector electromagnetic waves and scatterers of arbitrary size.

The problem has been formulated using a self-consistent approach. This approach leads to equations governing the expectation value of the total field and exciting field which are quite general and can be used for scatterers of any shape or size. They are written in terms of the scattering properties of a single, isolated scatterer.

The problem of scattering by spheres has been considered in detail. The rigorous Mie theory of scattering by a sphere has been used. In the Born approximation, which is quite adequate in the case of weakly random media, the results show that the distribution of scatterers is equivalent to a modified homogeneous medium where the refractive index is a function of the size, density and electromagnetic properties of the spheres. When multiple scattering effects are taken into account, it is found that the modified medium can sustain more than one mode. A dispersion relation has been obtained which governs the refractive indices corresponding to these

modes. For normal incidence, each of these modes is linearly polarized with a polarization similar to that of the incident wave.

The results obtained in this investigation reduce to those obtained by other authors when the special case of small spheres is considered. For instance, the Born approximation results lead to the well-known refractive index of the Rayleigh scattering theory. The basic techniques developed in this investigation can be used for further studies of scattering by spheres of arbitrary size and properties.

1. Introduction

A random medium can be defined as a medium some properties of which are random functions of position or time or both. Such a definition obviously includes almost all physical media due to its generality. However, since only macroscopic quantities can be measured experimentally in most cases, we usually assume that the medium can be treated as a continuum. Such an assumption requires a microscopic examination for its justification. The continuum theory has been successful for a large class of physical problems and, due to its simplicity, its use is very desirable as long as it is valid. There is, however, also a large class of problems that cannot be described by a simple continuum. For example, the randomness may be on a macroscopic scale and accessible to measurements. We shall, therefore, alternately define a random medium as a medium of which randomness is a salient feature. Examples of such randomness are the fluctuations in density in the troposphere and the ionosphere due to turbulence or other perturbing agencies, the airplane structure under random stresses excited by jet noise, and other similar phenomena.

The study of the propagation of electromagnetic waves in random media is interesting both theoretically and from the experimental point of view. Experimental studies have been greatly stimulated by the fact that electromagnetic waves can be used to study the medium itself. When the properties of a medium do not depart appreciably from the average value, the medium is said to be weakly random. In such cases a perturbation technique, such as the well known Born solution, can usually be used in theoretical investigations. If, however, the properties of the medium are allowed to change appreciably in some manner, the perturbation technique is largely useless and some new approach must be used. The present investigation goes from the weakly random to the strongly random media and hence both the perturbation method and the more exact formulation are used.

This thesis is concerned with the propagation of electromagnetic waves in a continuum in which are embedded randomly positioned, identical scatterers with similar orientation. The value of the electric field for a given configuration of scatterers is not usually of interest. We are more interested in the statistical expectation of the field for all the configurations of the ensemble. The positions of these scatterers are governed by the joint probability density function. It is assumed that the scattering properties of each individual scatterer are known. For a given configuration of scatterers, the total field at a point is given, according to the self-consistent approach, by the sum of the incident field and the fields scattered from all the scatterers. Therefore the total field depends upon the knowledge of the exciting fields at the scatterers. A similar self-consistent approach can be used to write equations for the exciting fields. In principle, these equations are to be solved to get the total field for a particular configuration. The ensemble average of the total field would then give the expectation value of the total field. Unfortunately, these equations are extremely complicated in practice and it is impossible to solve them directly. We are, therefore, forced to resort to an alternate route.

The alternate route is to average the equations as they stand. In so doing, we obtain a system of equations. The first equation involves the average total field and the first partial average of the exciting field. The first partial average of the exciting field on a scatterer is the average over all configurations of all other scatterers with this particular scatterer held fixed. The second equation involves the first partial average of the exciting field and the second partial average of the exciting field, which is the average taken with two scatterers held fixed. The third equation involves the second partial average and the third partial average and so on. Thus, we get a hierarchy of equations for the partial averages of the exciting field, each equation involving the partial average of one higher order. Since this chain of equations is not closed it is still impossible to solve, unless the chain can be broken by introducing valid approximations. These approximations and criteria of their validity are discussed by Foldy [1945], Lax [1951] and Waterman and Truell [1961]. Here we approximate the exciting field on a scatterer by the total field there when that scatterer is removed. It should be noted that this approximation is still much better than the single scattering approximation. Using this approximation, the average exciting field equation is closed and becomes a genuine integral equation which, if solved, determines the total field. The present investigation generalizes past work in two directions: the finite size scatterers and the vector nature of the field. The formulation is based on the work of Waterman and Truell [1961].

The special case of spherical scatters is next considered in detail. We consider a perfectly random distribution so that the joint probability distribution is equal to the product of the individual probabilities. Furthermore, we consider a constant density of, say, ρ_0 scatterers, per unit volume confined to half-space. The exact solution of scattering of a linearly polarized wave by a single sphere was obtained by Mie [1908]. This is used in first obtaining expressions for total average field in the Born approximation. This approximation is essentially the first order iteration of the exciting field equation in which all scatterers are assumed to be excited by the incident field alone. Physically this would be expected in cases of sparse concentration when the average separation of scatterers is large compared to their size and the wavelength. Techniques are developed to carry out integrations involving spherical vector wave functions. The results indicate that the polarization of the incident field is maintained and that, for cases where Born approximation would be expected to hold, the medium with scatterers behaves like an equivalent homogeneous medium with a modified propagation constant.

In strongly random media, the Born approximation is not valid and effects of multiple scattering have to be taken into account. For this purpose the integral equation governing the exciting field is considered. The geometry of the problem suggests that the exciting field will be linearly polarized having polarization similar to that of the incident field. We have made use of the "two-exterior" formalism of Twersky [1962a] to obtain a dispersion relation which determines the refractive index of the equivalent medium. Within the framework of the approximations mentioned earlier, this relation is valid for spheres of arbitrary size and electromagnetic properties (since all orders of multipoles in the Mie series have been taken into account) and for all orders of scattering.

The layout of the thesis is as follows: A brief historical survey of scattering problems and multiple scattering techniques is given in

Chapter 2. The problem is formulated in Chapter 3 and equations governing the quantities of interest are derived. The approximations involved are also discussed. Chapter 4 considers the Born approximation. The total field is derived by integrating the Mie series. In order not to digress from the main theme, the mathematical techniques developed for use in this integration are treated separately in an appendix. The problem of multiple scattering is next considered in Chapter 5. A dispersion relation governing the refractive index of the equivalent medium is derived. The new feature is that because of spatial dispersive effect many modes can propagate in the equivalent medium. The extinction theorem is shown to hold true. Chapter 6 deals with special cases. The equivalence of the Born approximation and the multiple scattering approach are shown for the case of small, perfectly conducting spheres of sparse concentration. These limiting results agree with those derived by other authors. The closing Chapter 7 discusses the conclusions arrived at from this research and indicates the directions in which this work is to be extended in the future.

2. Historical Survey

The problem of wave propagation in a medium containing a distribution of obstacles has been studied extensively due to its practical importance. The earliest studies were concerned with light and acoustic waves. Maxwell's work in the nineteenth century led to the identification of light as a form of electromagnetic radiation and laid the foundations for the modern approach to the subject. The first work on distributions of distinct objects was the development of the Lorentz-Lorenz formula in 1881 for the refractive index of most substances, plasmas excepted [Born and Wolf, 1959]. This was followed by Lord Rayleigh's classical work in 1899 on scattering by random distributions which explained the color of the sky. Scattering by single objects was also studied extensively following the development of coordinate systems in which the wave equation is separable. The problem of scattering by a sphere was solved by Gustav Mie in 1908 in terms of spherical vector wave functions. Extensive computations for scattering by single objects have recently been carried out at The University of Michigan using computers. A comprehensive review of the subject, with nearly 300 references, has been given by Twersky [1960].

In recent years, statistical methods have come to play an important part in the study of propagation in random media. The scintillation of radio signals received from radio stars and artificial earth satellites has added stimulus to this study. The statistical properties of these signals as effected by the fluctuations of density and refractive index of the medium have been studied by numerous workers such as Booker [1956], Chernov [1960], Keller [1962] and Yeh [1962] to mention but a few. The same problem can be treated from the point of view of a distribution of discrete scatterers in a homogeneous medium. Regular, periodic distributions have been studied as boundary value problems using Fourier analysis. This is not feasible for random distributions.

Foldy's work in 1945 was the first systematic treatment of multiple scattering of waves by a random distribution of point isotropic scatterers. He used the self-consistent approach to obtain expressions for the expectation values of the coherent and incoherent fields. His work was later extended by Lax [1951, 1952] to include anisotropic scatterers and inelastic scattering. Lax also considered the case when the scatterers are partially or completely ordered. One of the main difficulties in studying multi-scatterer problems lies in the estimation of the exciting field on a scatterer which is part of a configuration of scatterers. Various approximations made in this connection are discussed by Foldy and Lax. In a comprehensive paper on multiple scattering, Waterman and Truell [1961] have derived a criterion for the validity of these approximations. Their formulation of the prob... for scalar waves forms the basis of the formulation for vector waves used in the present work. Multiple scattering by sparse concentrations of small scatterers has also been treated by Twersky [1962a, b, c]. He has introduced the "two-exterior" formalism in which the exciting field and scattered field satisfy different wave equations. Some of his results have been derived as a special case of the present work.

Some experimental models have been built to simulate random distributions of spheres of various kinds. Measurements made by Twersky [1962] on simulated rare gas agree with his computations.

3. Formulation of the Problem

3.1 The Self-Consistent Field

Let us consider a collection of m identical scatterers of arbitrary size, shape and scattering properties, distributed randomly in the semi-infinite space $z \geq 0$.[1] Let the various configurations of scatterers be governed by the probability density distribution $p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$. Here $p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)\, dv_1\, dv_2 \ldots dv_m$ is the probability of finding the first scatterer in the volume dv_1 centered at $\underline{r}_1$, the second scatterer in dv_2 centered at $\underline{r}_2$ and so on. Since all scatterers are identical, a configuration is specified by the scatterer positions alone. We shall place two restrictions on this distribution:

(i) The scatterers are confined to the right half space. Therefore, $p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \equiv 0$ whenever any position vector $\underline{r}_j$ lies in the space $z < 0$.

(ii) Interpenetration of scatterers is excluded. Therefore, $p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \equiv 0$ whenever any two position vectors $\underline{r}_j$, $\underline{r}_k$ are such that the scatterers centered at $\underline{r}_j$ and $\underline{r}_k$ will overlap.

In addition, only elastic scattering will be considered. It is assumed that the scatterers are in no way effected by the incident field and that the motion of scatterers, if any, is too slow to be of significance.

Let an electromagnetic field $\underline{E}^i(\underline{r}, t)$ be incident from the left.

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[1] In the formulation it is not necessary to have this restriction. Actually scatterers can be anywhere. To be specific we shall assume they are restricted to the semi-infinite half-space.

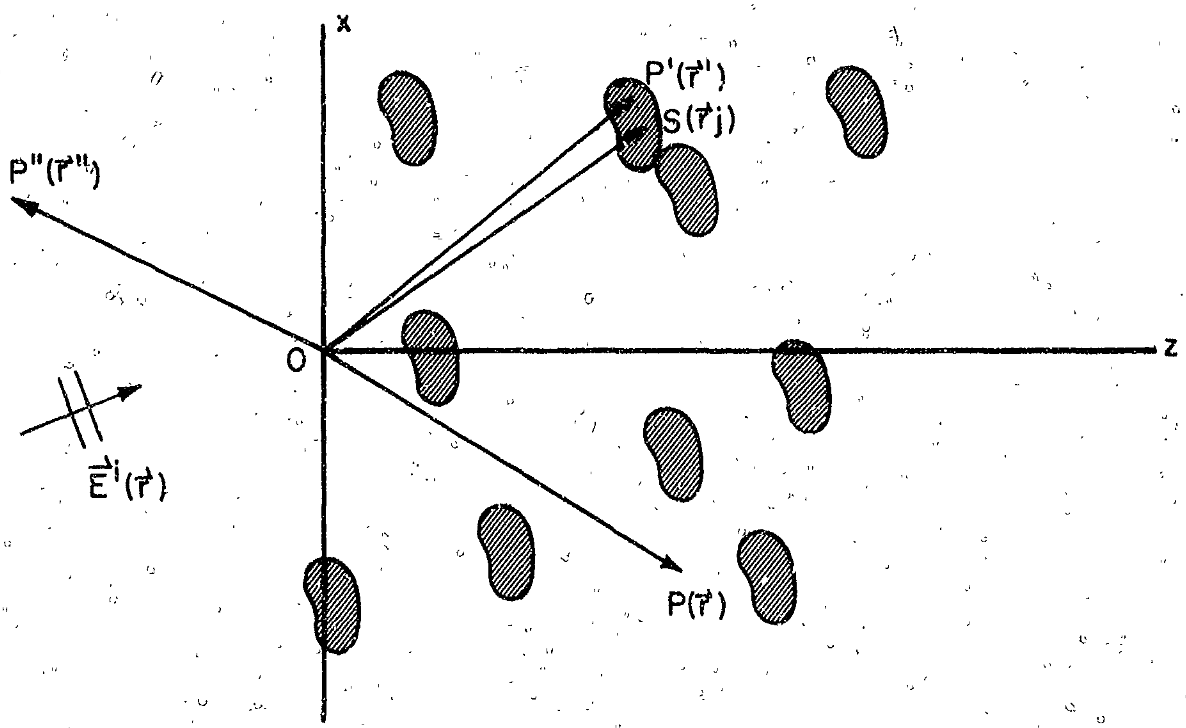

Figure 1. The geometry of the problem

We shall consider only the forced oscillation case with time dependence $e^{-i\omega t}$. For simplicity we shall usually suppress the time dependence. Our object is to find the total field at a point $\underline{r}$. For the configuration $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m$, we shall denote the total field at $\underline{r}$ by $\underline{E}(\underline{r}; \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$. Clearly, if $\underline{r}$ lies in $z \geqq 0$, it may lie outside all scatterers (as at P in Figure 1) or it may lie within some scatterer at $\underline{r}_j$ (as at P'). However, if $\underline{r}$ lies in $z < 0$ (as at P"), it must lie outside all scatterers[2]. We shall, therefore, consider the two cases separately.

3.11 Point of Observation Inside the Scattering Medium

Let $\underline{E}^s(\underline{r}, \underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$ denote the scattered field at $\underline{r}$ from the scatterer at $\underline{r}_j$ for a configuration with the first scatterer at $\underline{r}_1$, the second at $\underline{r}_2$, etc. This is governed by the exciting field at the scatterer at $\underline{r}_j$, denoted by $\underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$ and by the scattering properties of the scatterer which we shall denote by the operator $T(\underline{r}, \underline{r}_j)$. This operator operates on the exciting field at the scatterer at $\underline{r}_j$ to give the scattered field at $\underline{r}$. We thus have

$$\underline{E}^s(\underline{r}, \underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) = T(\underline{r}, \underline{r}_j)\, \underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) .$$

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[2] For convenience, we shall not consider the case when $\underline{r}$ lies in $z < 0$ but so close to the boundary that it may be within some scatterer which has its center within $z \geqq 0$. We shall, therefore, restrict $P(x, y, z)$ to lie outside the slab $-a \leqq z \leqq a$ where $2a$ is the effective dimension of the scatterers.

In this formulation we shall assume that the scattering properties of a single scatterer when isolated, are known so that $T(\underline{r}, \underline{r}_j)$ is known. We shall denote the total field at $\underline{r}$, when $\underline{r}$ is inside the scatterer at $\underline{r}_j$, by $T^I(\underline{r}, \underline{r}_j)\, \underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$ and shall assume, likewise, that $T^I(\underline{r}, \underline{r}_j)$, the interior scattering operator, is known. We shall further take $T(\underline{r}, \underline{r}_j) \equiv 0$ whenever $\underline{r}$ is inside the scatterer at $\underline{r}_j$ and $T^I(\underline{r}, \underline{r}_j) \equiv 0$ whenever $\underline{r}$ is outside the scatterer at $\underline{r}_j$. The total field at $\underline{r}$ for a fixed configuration $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m$ of scatterers is given by the sum of the incident field $\underline{E}^i(\underline{r})$ and the scattered fields from all m scatterers

$$\underline{E}(\underline{r} : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) = \underline{E}^i(\underline{r}) + \sum_{j=1}^{m} T(\underline{r}, \underline{r}_j)\, \underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$$

when $\underline{r}$ is outside all scatterers. When $\underline{r}$ is inside the scatterer at $\underline{r}_j$, the total field is given by

$$\underline{E}(\underline{r} : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) = T^I(\underline{r}, \underline{r}_j)\, \underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$$

These two equations can be combined into one by the following device used by Waterman and Truell [1961]. Let us define the symbol $\alpha(\underline{r}, \underline{r}_k)$ as follows:

$$\alpha(\underline{r}, \underline{r}_k) = \begin{array}{l} 0 \text{ when } \underline{r} \text{ is inside the scatterer at } \underline{r}_k \\ 1 \text{ when } \underline{r} \text{ is outside the scatterer at } \underline{r}_k \end{array}$$

Using this symbol, we can write the total field as

$$\underline{E}(\underline{r}: \underline{r}_1, \underline{r}_2,\ldots,\underline{r}_m) = [\prod_{k=1}^{m} \alpha(\underline{r},\underline{r}_k)][\underline{E}^i(\underline{r}) + \sum_{j=1}^{m} T(\underline{r},\underline{r}_j)\, \underline{E}^E(\underline{r}_j: \underline{r}_1, \underline{r}_2,\ldots,\underline{r}_m)]$$

$$+ \sum_{k=1}^{m} [1 - \alpha(\underline{r}, \underline{r}_k)][T^I(\underline{r}, \underline{r}_k)\, \underline{E}^E(\underline{r}_k: \underline{r}_1, \underline{r}_2,\ldots,\underline{r}_m)] \tag{1}$$

For a given type of scatterers, the total field cannot be evaluated for an arbitrary configuration. Therefore, we proceed to take the ensemble average of the equation governing the total field. The statistical expectation value of the total field (henceforth to be termed the average total field) is defined by

$$\langle\underline{E}(\underline{r})\rangle = \int dv_1 \int dv_2 \ldots \int dv_m\, p(\underline{r}_1, \underline{r}_2,\ldots,\underline{r}_m)\, \underline{E}(\underline{r}: \underline{r}_1, \underline{r}_2,\ldots,\underline{r}_m)$$

Each integration is carried out over the whole volume accessible to the scatterers. We get, from equation (1)

$$\langle\underline{E}(\underline{r})\rangle = \int dv_1 \int dv_2 \ldots \int dv_m\, p(\underline{r}_1, \underline{r}_2,\ldots,\underline{r}_m)[\prod_{k=1}^{m} \alpha(\underline{r}, \underline{r}_k)\, \underline{E}^i(\underline{r})]$$

$$+ \int dv_1 \int dv_2 \ldots \int dv_m\, p(\underline{r}_1\ldots\underline{r}_m)[\prod_{k=1}^{m} \alpha(\underline{r}, \underline{r}_k)][\sum_{j=1}^{m} T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)]$$

$$+ \int dv_1 \int dv_2 \ldots \int dv_m\, p(\underline{r}_1\ldots\underline{r}_m)[\sum_{k=1}^{m} \{1-\alpha(\underline{r},\underline{r}_k)\}\, T^I(\underline{r},\underline{r}_k)\underline{E}^E(\underline{r}_k:\underline{r}_1,\ldots,\underline{r}_m)] \tag{2}$$

There are three terms on the right of (2), and we shall simplify these terms one by one. In the first term, $\underline{E}^i(\underline{r})$ is independent of scatterer positions and, therefore, can be taken out of the integrations. Also, it has been shown by Waterman and Truell [1961] that, due to the exclusion of interpenetration, we can write

$$p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \prod_{k=1}^{m} \alpha(\underline{r}, \underline{r}_k) = p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)[1 - \sum_{k=1}^{m} \{1 - \alpha(\underline{r}, \underline{r}_k)\}] \tag{3}$$

Therefore, the first term becomes

$$\underline{E}^i(\underline{r}) \int dv_1 \int dv_2 \ldots \int dv_m \, p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) [1 - \sum_{k=1}^{m} \{1 - \alpha(\underline{r}, \underline{r}_k)\}]$$

Now the joint probability density can be written as

$$p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \, dv_1 \, dv_2, \ldots, dv_m = [p(\underline{r}_k) \, dv_k][p(\underline{r}_1, \underline{r}_2, \overset{\prime}{\ldots}, \underline{r}_m : \underline{r}_k) \, dv_1 \, dv_2 \overset{\prime}{\ldots} dv_m]$$

where $p(\underline{r}_1, \underline{r}_2, \overset{\prime}{\ldots}, \underline{r}_m : \underline{r}_k)$ is the conditional probability of finding scatterers at $\underline{r}_1, \underline{r}_2, \overset{\prime}{\ldots}, \underline{r}_m$, given a scatterer at $\underline{r}_k$. The prime is used to indicate that $\underline{r}_k$ is excluded from $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m$. The first term, therefore, reduces to

$$\underline{E}^i(\underline{r}) \int dv_1 \int dv_2 \ldots \int dv_m \, p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$$

$$- \underline{E}^i(\underline{r}) \sum_{k=1}^{m} \int dv_k \, p(\underline{r}_k) \{1 - \alpha(\underline{r}, \underline{r}_k)\} \int dv_1 \int dv_2 \overset{\prime}{\ldots} \int dv_m \, p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m : \underline{r}_k)$$

$$= \underline{E}^i(\underline{r})[1 - \sum_{k=1}^{m} \int dv_k \, p(\underline{r}_k) \{1 - \alpha(\underline{r}, \underline{r}_k)\}]$$

Here we have utilized the fact that the joint probability density and the conditional probability density are normalized to unity and, therefore,

$$\int dv_1 \int dv_2 \ldots \int dv_m \; p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) = 1$$

and
$$\int dv_1 \int dv_2 \ldots \int dv_m \; p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m : \underline{r}_k) = 1$$

Now the factor $[1-\alpha(\underline{r},\underline{r}_k)]$ in the remaining integral merely restricts the domain of integration to that region where $\underline{r}$ lies inside the scatterer at $\underline{r}_k$. Also since all scatterers are identical, the summation can be replaced by m times one integral. The single scatterer probability is given by

$$p(\underline{r}_k) = \frac{\rho(\underline{r}_k)}{m}$$

where $\rho(\underline{r}_k)$ is the number density of scatterers as a function of position. So the first term of (2) has the final form

$$\underline{E}^i(\underline{r}) \left[1 - m \int_{|\underline{r}-\underline{r}_k| < a} dv_k \, \frac{\rho(\underline{r}_k)}{m}\right] = \underline{E}^i(\underline{r}) \left[1 - \int_{|\underline{r}-\underline{r}'| < a} dv' \, \rho(\underline{r}')\right] \qquad (2a)$$

where $|\underline{r} - \underline{r}'| < a$ indicates that the domain of integration is such that $\underline{r}$ lies inside the scatterer centered at $\underline{r}'$.

We now consider the second term of equation (2). Using (3) we have

$$\sum_{j=1}^{m} \int dv_1 \int dv_2 \ldots \int dv_m \; p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \; T(\underline{r}, \underline{r}_j) \; \underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$$

$$- \int dv_1 \int dv_2 \ldots \int dv_m p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) [\sum_{k=1}^{m} \{1 - \alpha(\underline{r}, \underline{r}_k)\}][\sum_{j=1}^{m} T(\underline{r}, \underline{r}_j) \underline{E}^E(\underline{r}_j : \underline{r}_1, \ldots, \underline{r}_m)]$$

$$= \sum_{j=1}^{m} \int dv_j \; p(\underline{r}_j) \int dv_1 \overset{\prime}{\ldots} \int dv_m \; p(\underline{r}_1, \overset{\prime}{\ldots}, \underline{r}_m : \underline{r}_j) T(\underline{r}, \underline{r}_j) \underline{E}^E(\underline{r}_j : \underline{r}_1, \ldots, \underline{r}_m)$$

$$- \sum_{k=1}^{m} \sum_{j=1}^{m} \int dv_1 \int dv_2 \ldots \int dv_m p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) [1 - \alpha(\underline{r}, \underline{r}_k)] T(\underline{r}, \underline{r}_j) \underline{E}^E(\underline{r}_j : \underline{r}_1, \ldots, \underline{r}_m)$$

Now, in the first part,

$$\int dv_1 \overset{\prime}{\ldots} \int dv_m \; p(\underline{r}_1, \overset{\prime}{\ldots}, \underline{r}_m : \underline{r}_j) T(\underline{r}, \underline{r}_j) \underline{E}^E(\underline{r}_j : \underline{r}_1, \ldots, \underline{r}_m) = T(\underline{r}, \underline{r}_j) \langle \underline{E}^E(\underline{r}_j : \underline{r}_j) \rangle$$

where $\langle \underline{E}^E(\underline{r}_j : \underline{r}_j) \rangle$ is the first partial average of the exciting field on the scatterer at $\underline{r}_j$, averaged with this scatterer held fixed. Again since the scatterers are all identical, all terms in the summation are equal and the sum is equal to m times the single integral. Putting

$p(\underline{r}_j) = \dfrac{\rho(\underline{r}_j)}{m}$, the first part of the above expression becomes

$$\int_{|\underline{r}-\underline{r}'|>a} dv' \; \rho(\underline{r}') \; T(\underline{r}, \underline{r}') \; \langle \underline{E}^E(\underline{r}' : \underline{r}') \rangle$$

where the region of integration is limited by the condition that $T(\underline{r}, \underline{r}') = 0$ whenever $\underline{r}$ is inside the scatterer at $\underline{r}'$. So $\underline{r}'$ can take only those positions in which $\underline{r}$ is outside the scatterer at $\underline{r}'$. The second part of the expression has m^2 terms due to the double summation. Of these m^2 terms, m terms involve integrands of the type $[1-\alpha(\underline{r},\underline{r}_j)]T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)$, and the remaining (m^2-m) terms have integrands of the type $[1-\alpha(\underline{r},\underline{r}_k)]T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)$ with $j \neq k$. The first type can be treated as follows:

$$\int dv_1 \int dv_2 \ldots \int dv_m \; p(\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m)[1-\alpha(\underline{r},\underline{r}_j)]T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)$$

$$= \int dv_j \; p(\underline{r}_j)[1-\alpha(\underline{r},\underline{r}_j)] \int dv_1 \overset{'}{\ldots} \int dv_m p(\underline{r}_1,\overset{'}{\ldots},\underline{r}_m:\underline{r}_j)T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)$$

$$= \int dv_j \; p(\underline{r}_j) \; [1-\alpha(\underline{r},\underline{r}_j)] \; T(\underline{r},\underline{r}_j) \quad \langle \underline{E}^E(\underline{r}_j : \underline{r}_j)\rangle$$

$$= \; 0 \; ,$$

since, when $\underline{r}$ is inside the scatterer at $\underline{r}_j$, $T(\underline{r},\underline{r}_j) = 0$ and when $\underline{r}$ is outside the scatterer at $\underline{r}_j$, $[1-\alpha(\underline{r},\underline{r}_j)] = 0$. For the $(m^2 - m)$ terms of the second type we write the joint probability density as below:

$$p(\underline{r}_1, \underline{r}_2,\ldots,\underline{r}_m) = p(\underline{r}_j)\; p(\underline{r}_k : \underline{r}_j)\; p(\underline{r}_1,\overset{''}{\ldots},\underline{r}_m : \underline{r}_j, \underline{r}_k)$$

where the two primes in the last factor indicate that $\underline{r}_j$ and $\underline{r}_k$ are to be excluded from $\underline{r}_1,\ldots,\underline{r}_m$. Then these $(m^2 - m)$ terms become

$$\int dv_1 \int dv_2 \ldots \int dv_m \; p(\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m)[1-\alpha(\underline{r},\underline{r}_k)]T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)$$

$$= \int dv_j p(\underline{r}_j) \int dv_k p(\underline{r}_k:\underline{r}_j)[1-\alpha(\underline{r},\underline{r}_k)] \int dv_1 \ddot{\ldots} \int dv_m [p(\underline{r}_1,\ddot{\ldots},\underline{r}_m:\underline{r}_j,\underline{r}_k) \cdot$$

$$T(\underline{r},\underline{r}_j)\; \underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)]$$

$$= \int dv_j \; p(\underline{r}_j) \int dv_k \; p(\underline{r}_k:\underline{r}_j) \; [1-\alpha(\underline{r},\underline{r}_k)] \; T(\underline{r},\underline{r}_j) \; \langle \underline{E}^E(\underline{r}_j:\; \underline{r}_j,\; \underline{r}_k)\rangle$$

$$= \int\limits_{|\underline{r}-\underline{r}_j|>a} dv_j \; p(\underline{r}_j) \int\limits_{\substack{|\underline{r}-\underline{r}_k|<a \\ |\underline{r}_j-\underline{r}_k|>2a}} dv_k \; p(\underline{r}_k:\; \underline{r}_j) \; T(\underline{r},\underline{r}_j) \; \langle \underline{E}^E(\underline{r}_j:\; \underline{r}_j,\; \underline{r}_k)\rangle$$

where $|\underline{r}-\underline{r}_j| > a$ indicates that $\underline{r}$ should always be outside the scatterer at $\underline{r}_j$ (otherwise $T(\underline{r},\underline{r}_j) = 0$), $|\underline{r}-\underline{r}_k| < a$ indicates that $\underline{r}$ must be inside the scatterer at $\underline{r}_k$ (otherwise $[1-\alpha(\underline{r},\underline{r}_k)] = 0$) and $|\underline{r}_j-\underline{r}_k| > 2a$ indicates that the scatterer at $\underline{r}_j$ must be outside the scatterer at $\underline{r}_k$ (otherwise $p(\underline{r}_k:\underline{r}_j) = 0$). Now $p(\underline{r}_j) = \dfrac{\rho(\underline{r}_j)}{m}$ and $p(\underline{r}_k:\underline{r}_j) = \dfrac{\rho(\underline{r}_k:\underline{r}_j)}{m-1}$. Using these relations and the fact that the scatterers are identical, the contribution of the (m^2-m) terms becomes

$$(m^2-m) \int\limits_{|\underline{r}-\underline{r}_j|>a} dv_j \frac{\rho(\underline{r}_j)}{m} \int\limits_{\substack{|\underline{r}-\underline{r}_k|<a \\ |\underline{r}_j-\underline{r}_k|>2a}} dv_k \frac{\rho(\underline{r}_k:\underline{r}_j)}{m-1} \; T(\underline{r},\underline{r}_j) \; \langle \underline{E}^E(\underline{r}_j:\; \underline{r}_j,\; \underline{r}_k)\rangle$$

$$= \int\limits_{|\underline{r}-\underline{r}'|>a} dv' \rho(\underline{r}') \int\limits_{\substack{|\underline{r}-\underline{r}''|<a \\ |\underline{r}'-\underline{r}''|>2a}} dv'' \rho(\underline{r}'':\underline{r}') \; T(\underline{r},\underline{r}') \; \langle \underline{E}^E(\underline{r}':\; \underline{r}',\underline{r}'')\rangle$$

where $\langle \underline{E}^E(\underline{r}' : \underline{r}', \underline{r}'')\rangle$ is the second partial average of the exciting field on the scatterer at $\underline{r}'$ taken with the scatterers at $\underline{r}'$ and $\underline{r}''$ held fixed. Combining the contributions of the two parts, the second term of equation (2) can be written as

$$\int_{|\underline{r}-\underline{r}'|>a} dv'\, \rho(\underline{r}')\, T(\underline{r},\underline{r}')\, \langle \underline{E}^E(\underline{r}' : \underline{r}')\rangle$$

$$- \int_{|\underline{r}-\underline{r}'|>a} dv'\, \rho(\underline{r}') \int_{\substack{|\underline{r}-\underline{r}''|<a \\ |\underline{r}'-\underline{r}''|>2a}} dv''\, \rho(\underline{r}'' : \underline{r}')\, T(\underline{r},\underline{r}')\, \langle \underline{E}^E(\underline{r}' : \underline{r}', \underline{r}'')\rangle \qquad (2b)$$

The third term of equation (2) can be simplified in a straightforward manner as follows:

$$\int dv_1 \int dv_2 \ldots \int dv_m\, p(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)\Big[\sum_{k=1}^{m} \{1-\alpha(\underline{r},\underline{r}_k)\} T^I(\underline{r},\underline{r}_k) \underline{E}^E(\underline{r}_k : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)\Big]$$

$$= \sum_{k=1}^{m} \int dv_k\, p(\underline{r}_k)\{1-\alpha(\underline{r},\underline{r}_k)\} \int dv_1 \ldots \int dv_m\, p(\underline{r}_1, \ldots, r_m : \underline{r}_k) T^I(\underline{r},\underline{r}_k) \underline{E}^E(\underline{r}_k : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$$

$$= \sum_{k=1}^{m} \int dv_k\, p(\underline{r}_k)\, \{1-\alpha(\underline{r},\underline{r}_k)\}\, T^I(\underline{r},\underline{r}_k)\, \langle \underline{E}^E(\underline{r}_k : \underline{r}_k)\rangle$$

$$= \int_{|\underline{r}-\underline{r}'|<a} dv'\, \rho(\underline{r}')\, T^I(\underline{r},\underline{r}')\, \langle \underline{E}^E(\underline{r}' : \underline{r}')\rangle \qquad (2c)$$

Here the domain of integration is limited by the fact that both $[1-\alpha(\underline{r},\underline{r}_k)]$ and $T^I(\underline{r},\underline{r}_k)$ are zero when $\underline{r}$ is outside the scatterer at $\underline{r}_k$. We have again used $p(\underline{r}_k) = \frac{\rho(\underline{r}_k)}{m}$ and replaced the sum by m times the integral since the scatterers are identical.

Putting the three expressions (2a), (2b) and (2c) together we get the required expression for the average total field

$$\langle \underline{E}(\underline{r}) \rangle = \underline{E}^i(\underline{r}) \left[1 - \int_{|\underline{r}-\underline{r}'|<a} dv' \rho(\underline{r}')\right] + \int_{|\underline{r}-\underline{r}'|>a} dv' \rho(\underline{r}') T(\underline{r},\underline{r}') \langle \underline{E}^E(\underline{r}':\underline{r}') \rangle$$

$$- \int_{|\underline{r}-\underline{r}'|>a} dv' \rho(\underline{r}') \int_{\substack{|\underline{r}-\underline{r}''|<a \\ |\underline{r}'-\underline{r}''|>2a}} dv'' \rho(\underline{r}'':\underline{r}') T(\underline{r},\underline{r}') \langle \underline{E}^E(\underline{r}':\underline{r}',\underline{r}'') \rangle$$

$$+ \int_{|\underline{r}-\underline{r}'|<a} dv' \rho(\underline{r}') T^I(\underline{r},\underline{r}') \langle \underline{E}^E(\underline{r}':\underline{r}') \rangle \qquad (4)$$

It may be noted in passing that if there are no scatterers in the matrix medium then (4) reduces to

$$\langle \underline{E}(\underline{r}) \rangle = \underline{E}^i(\underline{r})$$

as would be expected. On the other hand, if the number density of scatterers is so large that the entire right half space is filled with scatterers, equation (4) shows that the incident field is extinguished. This is because in this case the scatterer density is constant and we have

$$\int_{|\underline{r}-\underline{r}'|<a} dv' \rho(\underline{r}') = \rho_o \int_{|\underline{r}-\underline{r}'|<a} dv' = v_s$$

where v_s is the fractional volume occupied by the scatterers. When scatterers occupy the entire right half space, the fractional occupied volume is unity. Therefore, we have

$$\underline{E}^i(\underline{r})\,[1 - \int_{|\underline{r}-\underline{r}'|<a} dv'\,\rho(\underline{r}')] = 0$$

This is consistent with the extinction theorem. Actually the extinction theorem holds even when the fractional occupied volume is less than one. This will be discussed in Chapter 5.

3.12 Point of Observation Outside the Scattering Medium

When $\underline{r}$ lies in the space $z < 0$ (excluding the edge region as mentioned on page 10), it is always outside all scatterers and the total field equations can be easily derived as follows:

$$\underline{E}(\underline{r}:\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m) = \underline{E}^i(\underline{r}) + \sum_{j=1}^{m} T(\underline{r},\underline{r}_j)\,\underline{E}^E(\underline{r}_j:\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m)$$

Therefore, the average value is

$$\langle\underline{E}(\underline{r})\rangle = \int dv_1 \int dv_2 \ldots \int dv_m\, p(\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m)\,\underline{E}^i(\underline{r})$$

$$+ \int dv_1 \int dv_2 \ldots \int dv_m\, p(\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m)[\sum_{j=1}^{m} T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m)]$$

$$= \underline{E}^i(\underline{r}) + \sum_{j=1}^{m} \int dv_j\, p(\underline{r}_j) \int dv_1 \ldots \int dv_m\, p(\underline{r}_1,\ldots,\underline{r}_m:\underline{r}_j)T(\underline{r},\underline{r}_j)\underline{E}^E(\underline{r}_j:\underline{r}_1\ldots\underline{r}_m)$$

$$= \underline{E}^i(\underline{r}) + \sum_{j=1}^{m} \int dv_j\; p(\underline{r}_j)\; T(\underline{r},\underline{r}_j)\; \langle\underline{E}^E(\underline{r}_j:\underline{r}_j)\rangle$$

or

$$\langle \underline{E}(\underline{r}) \rangle = \underline{E}^i(\underline{r}) + \int_{|\underline{r}-\underline{r}'|>a} dv' \, \rho(\underline{r}') \, T(\underline{r},\underline{r}') \, \langle \underline{E}^E(\underline{r}' : \underline{r}') \rangle \tag{5}$$

This is in the form of a sum of the incident and the "reflected" fields.

3.13 The Exciting Field

The exciting field on a scatterer centered at $\underline{r}_1$ is given by the self-consistent equation

$$\underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) = \underline{E}^i(\underline{r}_1) + \sum_{j=2}^{m} T(\underline{r}_1, \underline{r}_j) \, \underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \tag{6}$$

To get the first partial average when the scatterer at $\underline{r}_1$ is held fixed, we use $p(\underline{r}_2, \underline{r}_3, \ldots, \underline{r}_m : \underline{r}_1)$ and integrate over all positions except $\underline{r}_1$. We get

$$\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1) \rangle = \int dv_2 \ldots \int dv_m \, p(\underline{r}_2, \ldots, \underline{r}_m : \underline{r}_1) \, \underline{E}^i(\underline{r}_1)$$

$$+ \int dv_2 \ldots \int dv_m \, p(\underline{r}_2, \ldots, \underline{r}_m : \underline{r}_1) \left[\sum_{j=2}^{m} T(\underline{r}_1, \underline{r}_j) \, \underline{E}^E(\underline{r}_j : \underline{r}_1, \ldots, \underline{r}_m) \right]$$

$$= \underline{E}^i(\underline{r}_1) + \sum_{j=2}^{m} \int dv_j \, p(\underline{r}_j : \underline{r}_1) \int dv_2 \ldots \int dv_m \, p(\underline{r}_2, \ldots, \underline{r}_m : \underline{r}_1, \underline{r}_j) T(\underline{r}_1, \underline{r}_j) \underline{E}^E(\underline{r}_j : \underline{r}_1 \ldots \underline{r}_m)$$

$$= \underline{E}^i(\underline{r}_1) + \sum_{j=2}^{m} \int dv_j \, p(\underline{r}_j : \underline{r}_1) \, T(\underline{r}_1, \underline{r}_j) \, \langle \underline{E}^E(\underline{r}_j : \underline{r}_j, \underline{r}_1) \rangle$$

or

$$\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1) \rangle = \underline{E}^i(\underline{r}_1) + \int_{|\underline{r}_1-\underline{r}'|>2a} dv' \, \rho(\underline{r}' : \underline{r}_1) \, T(\underline{r}_1, \underline{r}') \, \langle \underline{E}^E(\underline{r}' : \underline{r}', \underline{r}_1) \rangle \tag{7}$$

We have put

$$p(\underline{r}_j : \underline{r}_1) = \frac{\rho(\underline{r}_j : \underline{r}_1)}{m-1}$$

and have replaced the sum of (m-1) terms by (m-1) times one term. The domain of integration, denoted by $|\underline{r}_1-\underline{r}'|>2a$, is such that the scatterer at $\underline{r}_1$ is outside the scatterer at $\underline{r}'$. This is governed by $\rho(\underline{r}':\underline{r}_1)$ which is zero outside this domain due to exclusion of interpenetration. We notice from (7) that the first partial average $\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1)\rangle$ is given in terms of the second partial average $\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}')\rangle$ of the exciting field. The second partial average of, say, the exciting field at $\underline{r}_1$ with scatterers at $\underline{r}_1$ and $\underline{r}_2$ held fixed, is obtained from (6) by multiplying by $p(\underline{r}_3, \ldots, \underline{r}_m : \underline{r}_1, \underline{r}_2)\, dv_3 \ldots dv_m$ and integrating. We get:

$$\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2)\rangle = \int dv_3 \int dv_4 \ldots \int dv_m\, p(\underline{r}_3 \ldots \underline{r}_m : \underline{r}_1, \underline{r}_2)[\underline{E}^i(\underline{r}_1) + \sum_{j=2}^{m} T(\underline{r}_1, \underline{r}_j)\underline{E}^E(\underline{r}_j : \underline{r}_1 \ldots \underline{r}_m)]$$

$$= \underline{E}^i(\underline{r}_1) + \int dv_3 \ldots \int dv_m\, p(\underline{r}_3 \ldots \underline{r}_m : \underline{r}_1, \underline{r}_2)\, T(\underline{r}_1, \underline{r}_2)\, \underline{E}^E(\underline{r}_2 : \underline{r}_1, \ldots, \underline{r}_m)$$

$$+ \sum_{j=3}^{m} \int dv_j\, p(\underline{r}_j : \underline{r}_1, \underline{r}_2) \int dv_3 \ldots \int dv_m\, p(\underline{r}_3 \ldots \underline{r}_m : \underline{r}_1, \underline{r}_2, \underline{r}_j) T(\underline{r}_1, \underline{r}_j) \underline{E}^E(\underline{r}_j : \underline{r}_1 \ldots \underline{r}_m)$$

$$= \underline{E}^i(\underline{r}_1) + T(\underline{r}_1, \underline{r}_2) \langle \underline{E}^E(\underline{r}_2 : \underline{r}_1, \underline{r}_2)\rangle$$

$$+ \sum_{j=3}^{m} \int dv_j \frac{\rho(\underline{r}_j : \underline{r}_1, \underline{r}_2)}{m-2}\, T(\underline{r}_1, \underline{r}_j) \langle \underline{E}^E(\underline{r}_j : \underline{r}_1, \underline{r}_2, \underline{r}_j)\rangle$$

or

$$\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2)\rangle = \underline{E}^i(\underline{r}_1) + T(\underline{r}_1, \underline{r}_2)\langle \underline{E}^E(\underline{r}_2 : \underline{r}_1, \underline{r}_2)\rangle + \int_{\substack{|\underline{r}_1-\underline{r}'|>2a \\ |\underline{r}_2-\underline{r}'|>2a}} dv'\, \rho(\underline{r}' : \underline{r}_1, \underline{r}_2) T(\underline{r}_1, \underline{r}') \langle \underline{E}^E(\underline{r}' : \underline{r}_1, \underline{r}_2, \underline{r}')\rangle \qquad (8)$$

Thus the second partial average is given by an equation involving the third partial average. It is obvious that this procedure can be repeated to get higher partial averages and we ultimately get m equations. The last one, in fact, will be the exciting field equation for a fixed configuration which is the average of equation (6) itself.

An alternate approach to the problem is the use of iteration technique. This technique is especially useful if the medium is weakly random. Equation (6) can be written in terms of successive orders of scattering by repeated iterations as below:

$$\begin{aligned}
\underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) = \underline{E}^i(\underline{r}_1) &+ \sum_{j=2}^{m} T(\underline{r}_1, \underline{r}_j)\, \underline{E}^i(\underline{r}_j) \\
&+ \sum_{j=2}^{m} T(\underline{r}_1, \underline{r}_j) \Big[\sum_{\substack{k=1 \\ \neq j}}^{m} T(\underline{r}_j, \underline{r}_k)\, \underline{E}^i(\underline{r}_k) \Big] \\
&+ \sum_{j=2}^{m} T(\underline{r}_1, \underline{r}_j) \Big[\sum_{\substack{k=1 \\ \neq j}}^{m} T(\underline{r}_j, \underline{r}_k) \Big\{ \sum_{\substack{\ell=1 \\ \neq k}}^{m} T(\underline{r}_k, \underline{r}_\ell)\, \underline{E}^i(\underline{r}_\ell) \Big\} \Big] \\
&+ \cdots
\end{aligned} \tag{9}$$

If either (6) or (9) could be solved, the result could be substituted in the equation for the (m-1)st partial average and, hopefully, we could solve that equation. By successive solutions and substitutions we could ultimately solve the equation for the first partial average. In practice this is impossible due to the large number of scatterers involved in multiple scattering situations. Some approximations are, therefore, necessary.

3.2 Approximations in Multiple Scattering

In order to formulate the many-body scattering problem in a form that can be solved for specific cases, we have to consider some approximations.

One approach is to look at multiple scattering from the point of view of successive orders of scattering as expressed in equation (9). The first approximation would be to consider the first term alone and replace the exciting field by the incident field itself. This is called the Born approximation. It has been used in Chapter 4 to solve the problem when the scatterers are spherical in shape. This approximation is good enough when the average separation of scatterers is large compared to their size. In the second approximation, each scatterer would be excited by the incident field plus the once-scattered field. Such successive approximations can be made to get results to any desired accuracy if one can evaluate the integrals involved. In the case of spheres it has not been possible to evaluate the integrals involved in the second approximation.

Another approach is to consider the exciting field at a scatterer at $\underline{r}_j$ in a given configuration as an expansion in which the first term is the total field at $\underline{r}_j$ when this scatterer is not there (that is, in a configuration of (m-1) scatterers). The second and higher terms then include the rescattering of the field scattered from this scatterer when it is put back in the configuration. Thus we have

$$\underline{E}^E(\underline{r}_j:\underline{r}_1,\ldots,\underline{r}_m) = \underline{E}(\underline{r}_j:\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m)$$

$$+ \sum_{\substack{k=1\\ \neq j}}^{m} T(\underline{r}_j,\underline{r}_k)\left\{T(\underline{r}_k,\underline{r}_j)\ \underline{E}(\underline{r}_j:\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m)\right\}$$

$$+ \ldots \qquad (10)$$

The approximation consists in neglecting the second and higher terms on the right hand side. For dense systems in which multiple scattering effects are most important, this is a much better approximation than the Born approximation. A comparison of the magnitude of the second term with that of the first has been made by Waterman and Truell [1961] by considering point scatterers and scalar waves. They have developed a criterion according to which the second term is much smaller than the first if

$$\frac{\rho_o Q_s}{k} << 1$$

where ρ_o is the number density of scatterers (assumed constant), Q_s is the scattering cross section of a single scatterer and k is the propagation constant of the medium in which the scatterers are located. This criterion is shown to be quite generally valid for most physical situations.

A third approach is to consider the hierarchy of equations for partial averages of which equations (7) and (8) for $\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1)\rangle$ and $\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2)\rangle$ are the first two. The approximation consists in breaking the hierarchy at some point, that is, taking

$$\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_i, \underline{r}_j)\rangle \approx \langle \underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_i)\rangle$$

for some i and j. If we break the hierarchy at the first equation itself then we have

$$\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2)\rangle \approx \langle \underline{E}^E(\underline{r}_1 : \underline{r}_1)\rangle \qquad (11)$$

This approximation has been discussed by Lax [1952] and is designated as the "quasi-crystalline" approximation, since, in the case of crystals, it holds exactly. He has shown that it is a very good approximation in the case of dense systems where multiple scattering effects are most important. It is equivalent to neglecting the fluctuations in the exciting field at $\underline{r}_1$ due to the fluctuation of the scatterer at $\underline{r}_2$ about its mean position.

We shall use the last two approaches to simplify our equations.

3.3 Approximate Equations

Let us approximate the exciting field at the scatterer at $\underline{r}_1$ by the total field at $\underline{r}_1$ when the scatterer at $\underline{r}_1$ is removed from the configuration. We get:

$$\underline{E}^E(\underline{r}_1 : \underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \approx \underline{E}(\underline{r}_1 : \underline{r}_2, \underline{r}_3, \ldots, \underline{r}_m)$$

$$= \underline{E}^i(\underline{r}_1) + \sum_{j=2}^{m} T(\underline{r}_1, \underline{r}_j)\, \underline{E}^E(\underline{r}_j : \underline{r}_2, \ldots, \underline{r}_m)$$

Multiplying both sides by $p(\underline{r}_2, r_3, \ldots, \underline{r}_m : \underline{r}_1)\, dv_2 \ldots dv_m$ and integrating we get

$$\langle \underline{E}^E(\underline{r}_1 : \underline{r}_1) \rangle = \underline{E}^i(\underline{r}_1) + \sum_{j=2}^{m} \int dv_2 \ldots dv_m\, p(\underline{r}_2, \ldots, \underline{r}_m : \underline{r}_1) T(\underline{r}_1, \underline{r}_j) \underline{E}^E(\underline{r}_j : \underline{r}_2, \ldots, \underline{r}_m)$$

Now let us write

$$p(\underline{r}_2, \ldots, \underline{r}_m : \underline{r}_1) = p(\underline{r}_j : \underline{r}_1)\, p(\underline{r}_2, \overset{\prime}{\ldots}, \underline{r}_m : \underline{r}_j)$$

$$- p(\underline{r}_j : \underline{r}_1)[p(\underline{r}_2, \overset{\prime}{\ldots}, \underline{r}_m : \underline{r}_j) - p(\underline{r}_2, \overset{\prime}{\ldots}, \underline{r}_m : \underline{r}_j, \underline{r}_1)]$$

Using this expression we get

$$\langle \underline{E}^E(\underline{r}_1;\underline{r}_1)\rangle = \underline{E}^i(\underline{r}_1)$$

$$+ \sum_{j=2}^{m} \left[\int dv_j\, p(\underline{r}_j:\underline{r}_1) \int dv_2 \ldots \int dv_m\, p(\underline{r}_2 \ldots \underline{r}_m:\underline{r}_j) T(\underline{r}_1,\underline{r}_j) \underline{E}^E(\underline{r}_j:\underline{r}_2,\ldots,\underline{r}_m) \right] - R$$

$$= \underline{E}^i(\underline{r}_1) + \sum_{j=2}^{m} \left[\int dv_j\; p(\underline{r}_j:\underline{r}_1)\; T(\underline{r}_1,\underline{r}_j)\; \langle \underline{E}^E(\underline{r}_j:\underline{r}_j)\rangle_{m-1} \right] - R$$

$$= \underline{E}^i(\underline{r}_1) + \int_{|\underline{r}_1-\underline{r}'|>2a} dv'\; \rho(\underline{r}':\underline{r}_1)\; T(\underline{r}_1,\underline{r}')\; \langle \underline{E}^E(\underline{r}':\underline{r}')\rangle_{m-1} - R$$

Here the notation $\langle \underline{E}^E(\underline{r}':\underline{r}')\rangle_{m-1}$ indicates the first partial average of the exciting field at $\underline{r}'$ when the scatterer at $\underline{r}'$ is held fixed, taken over the ensemble of configurations of (m-1) scatterers. Obviously if the number of scatterers is very large, $\langle \underline{E}^E(\underline{r}':\underline{r}')\rangle_{m-1} \approx \langle \underline{E}^E(\underline{r}':\underline{r}')\rangle$. The term R is given by

$$R = \sum_{j=2}^{m} \int dv_2 \ldots \int dv_m\; p(\underline{r}_j:\underline{r}_1) [p(\underline{r}_2 \ldots \underline{r}_m:\underline{r}_j)$$

$$- p(\underline{r}_2,\ldots,\underline{r}_m:\underline{r}_j,\underline{r}_1)] T(\underline{r}_1,\underline{r}_j)\; \underline{E}^E(\underline{r}_j:\underline{r}_2,\ldots,\underline{r}_m)$$

In the case of perfectly random distributions, the scatterers are statistically independent and

$$p(\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_m) = p(\underline{r}_1)\; p(\underline{r}_2) \ldots p(\underline{r}_m)$$

In this case

$$[p(\underline{r}_2,\ldots,\underline{r}_m:\underline{r}_j) - p(\underline{r}_2,\ldots,\underline{r}_m:\underline{r}_1,\underline{r}_j)] = 0$$

and, therefore, R = 0. Also, $p(\underline{r}':\underline{r}_1) = p(\underline{r}')$. The domain of integration takes care of the exclusion of interpenetration. So, for the case when the number of scatterers is very large and the distribution is statistically independent we have

$$\langle \underline{E}^E(\underline{r}:\underline{r})\rangle = \underline{E}^i(\underline{r}) + \int_{|\underline{r}-\underline{r}'|>2a} dv'\,\rho(\underline{r}')\,T(\underline{r},\underline{r}')\,\langle \underline{E}^E(\underline{r}':\underline{r}')\rangle \qquad (13)$$

A comparison with equation (7) shows that for statistically independent scatterers, the approximation

$$\underline{E}^E(\underline{r}_1:\underline{r}_1,\ldots,\underline{r}_m) \approx \underline{E}(\underline{r}_1:\underline{r}_2,\ldots,\underline{r}_m)$$

is equivalent to the "quasi-crystalline" approximation

$$\langle \underline{E}^E(\underline{r}_1:\underline{r}_1,\underline{r}_2)\rangle \approx \langle \underline{E}^E(\underline{r}_1:\underline{r}_1)\rangle$$

We shall now use this approximation to simplify the total field equation.

When the point of observation is in the region $z \geq 0$ the average total field is given by equation (4) which, with the above approximations, becomes:

$$\langle \underline{E}(\underline{r})\rangle = \underline{E}^i(\underline{r})[1-\int_{|\underline{r}-\underline{r}'|<a} dv'\rho(\underline{r}')] + \int_{|\underline{r}-\underline{r}'|>a} dv'\rho(\underline{r}')T(\underline{r},\underline{r}')\langle \underline{E}^E(\underline{r}':\underline{r}')\rangle[1-\int_{\substack{|\underline{r}-\underline{r}''|<a \\ |\underline{r}'-\underline{r}''|>2a}} dv''\rho(\underline{r}'')]$$

$$+ \int_{|\underline{r}-\underline{r}'|<a} dv'\,\rho(\underline{r}')\,T^I(\underline{r},\underline{r}')\,\langle \underline{E}^E(\underline{r}':\underline{r}')\rangle$$

In the second term we have put

$$\int_{|\underline{r}-\underline{r}'|>a} dv'\, \rho(\underline{r}') \int_{\substack{|\underline{r}-\underline{r}''|<a \\ |\underline{r}'-\underline{r}''|>2a}} dv''\, \rho(\underline{r}'':\underline{r}')\, T(\underline{r},\underline{r}') \langle \underline{E}^{E}(\underline{r}:\underline{r}',\underline{r}'')\rangle$$

$$= \int_{|\underline{r}-\underline{r}'|>a} dv'\, \rho(\underline{r}')\, T(\underline{r},\underline{r}') \langle \underline{E}^{E}(\underline{r}':\underline{r}')\rangle \int_{\substack{|\underline{r}-\underline{r}''|<a \\ |\underline{r}'-\underline{r}''|>2a}} dv''\, \rho(\underline{r}'')$$

We note that except for the case when $\underline{r}'$ is near $\underline{r}$, the $\underline{r}''$-integration can be carried out over the domain $|\underline{r}-\underline{r}''|<a$ since no chance of overlapping of the scatterers at $\underline{r}'$ and $\underline{r}''$ will arise. Since the $\underline{r}'$-integration is over the entire half-space such that $\underline{r}$ is outside the scatterer at $\underline{r}'$ and the $\underline{r}''$-integration is over a small volume of the size of a single scatterer such that $\underline{r}$ is always within the scatterer at $\underline{r}''$, no significant error will be involved in replacing

$$\int_{\substack{|\underline{r}-\underline{r}''|<a \\ |\underline{r}'-\underline{r}''|>2a}} dv''\, \rho(\underline{r}'')$$

by

$$\int_{|\underline{r}-\underline{r}''|<a} dv''\, \rho(\underline{r}'')$$

Then the average total field at a point in the space $z \geq 0$ becomes

$$\langle \underline{E}(\underline{r})\rangle = [1 - \int\limits_{|\underline{r}-\underline{r}'|<a} dv' \, \rho(\underline{r}')][\underline{E}^i(\underline{r}) + \int\limits_{|\underline{r}-\underline{r}'|>a} dv' \, \rho(\underline{r}')T(\underline{r},\underline{r}')\langle \underline{E}^E(\underline{r}':\underline{r}')\rangle]$$

$$+ \int\limits_{|\underline{r}-\underline{r}'|<a} dv' \, \rho(\underline{r}') \, T^I(\underline{r},\underline{r}') \, \langle \underline{E}^E(\underline{r}':\underline{r}')\rangle \qquad (14)$$

When the point of observation is in the space $z < 0$, the average total field is given by equation (5).

We now have a complete formulation for the average total field both when the point of observation is inside the region $z \geq 0$ to which the scatterers are confined, and when it is outside this region. The total field is given in terms of the operators $T(\underline{r},\underline{r}')$ and $T^I(\underline{r},\underline{r}')$ and the first partial average of the exciting field. We shall now proceed to solve the problem for the case of spherical scatterers in the following chapters. For reference purposes, we recapitulate the relevant equations below:

$$\langle \underline{E}(\underline{r})\rangle = [1 - \int\limits_{|\underline{r}-\underline{r}'|<a} dv'\rho(\underline{r}')][\underline{E}^i(\underline{r}) + \int\limits_{|\underline{r}-\underline{r}'|>a} dv'\rho(\underline{r}') \, T(\underline{r},\underline{r}')\langle \underline{E}^E(\underline{r}':\underline{r}')\rangle]$$

$$+ \int\limits_{|\underline{r}-\underline{r}'|<a} dv'\rho(\underline{r}')T^I(\underline{r},\underline{r}') \, \langle \underline{E}^E(\underline{r}':\underline{r}')\rangle \quad \text{when } \underline{r} \text{ lies in } z \geq 0 \qquad (14)$$

$$\langle \underline{E}(\underline{r})\rangle = \underline{E}^i(\underline{r}) + \int\limits_{|\underline{r}-\underline{r}'|>a} dv'\rho(\underline{r}') \, T(\underline{r},\underline{r}') \, \langle \underline{E}^E(\underline{r}':\underline{r}')\rangle \quad \text{when } \underline{r} \text{ lies in } z < 0 \qquad (5)$$

where the exciting field in any region satisfies the equation

$$\langle \underline{E}^E(\underline{r}:\underline{r})\rangle = \underline{E}^i(\underline{r}) + \int\limits_{|\underline{r}-\underline{r}'|>2a} dv'\rho(\underline{r}') \, T(\underline{r},\underline{r}') \, \langle \underline{E}^E(\underline{r}':\underline{r}')\rangle \qquad (13)$$

4. Single Scattering by Spherical Scatterers

4.1 The Average Total Field Equation

It has been indicated earlier that we can evaluate the total field to any desired degree of accuracy by considering successive orders of scattering. Mathematical difficulties, however, make it impossible to obtain exact expressions for the total field even for the first order scattering for any but the simplest geometrical shapes of scatterers. In this chapter we shall consider scattering by spheres and shall consider the first order scattering only. This is called the Born approximation and consists in replacing the exciting field $\langle \underline{E}^E(\underline{r};\underline{r})\rangle$ by $\underline{E}^i(\underline{r})$ on the right-hand side of equation (13). Let the incident wave be a linearly polarized plane wave, incident normally, given by

$$\underline{E}^i(\underline{r}) = \hat{i}_x e^{ikz} \quad .$$

We shall consider the number density of spheres to be constant so that $\rho(\underline{r}') = \rho_o$ for $z \geqq 0$ and $\rho(\underline{r}') = 0$ for $z < 0$. The average total field at $\underline{r}(x,y,z)$ is, therefore, given by equations (14) and (5) which, for this case, can be written as

$$\langle \underline{E}(\underline{r})\rangle = \Big[1 - \rho_o \int\limits_{\substack{|\underline{r}-\underline{r}'|<a \\ z' \geqq 0}} dv'\Big]\Big[\underline{E}^i(\underline{r}) + \rho_o \int\limits_{\substack{|\underline{r}-\underline{r}'|>a \\ z' \geqq 0}} dv'\, T(\underline{r},\underline{r}')\, \underline{E}^i(\underline{r}')\Big]$$

$$+ \rho_o \int\limits_{\substack{|\underline{r}-\underline{r}'|<a \\ z' \geqq 0}} dv'\, T^I(\underline{r},\underline{r}')\, \underline{E}^i(\underline{r}') \qquad \text{for } z > a \qquad (15)$$

and

$$\langle \underline{E}(\underline{r})\rangle = \underline{E}^{i}(\underline{r}) + \rho_o \int_{z' \geqq 0} dv'\, T(\underline{r},\underline{r}')\, \underline{E}^{i}(\underline{r}') \quad \text{for } z < -a \qquad (16)$$

where a is the radius of the spheres. Since $\underline{E}^{i}(\underline{r})$ is a known quantity, we need to know $T(\underline{r},\underline{r}')\ \underline{E}^{i}(\underline{r}')$, which is the scattered field at $\underline{r}$ from a scatterer at $\underline{r}'$ excited by $\underline{E}^{i}(\underline{r}')$, and $T^{I}(\underline{r},\underline{r}')\ \underline{E}^{i}(\underline{r}')$, which is the field at a point $\underline{r}$ inside a scatterer at $\underline{r}'$ excited by $\underline{E}^{i}(\underline{r}')$. Knowing these expressions, we can attempt to carry out the integration and get $\langle \underline{E}(\underline{r})\rangle$. First of all, therefore, we need to study the scattering properties of an isolated sphere.

4.2 Scattering of Vector Waves by an Isolated Sphere

The problem of scattering of a linearly polarized wave by a sphere is solved in terms of an infinite series, usually called the Mie series after Gustav Mie, who first solved this problem in 1908. One seeks a solution of the vector wave equation

$$\nabla^2 \underline{A} + k^2 \underline{A} = 0$$

which will satisfy the boundary conditions on the surface of the sphere. It is found that solutions of the vector wave equation can be generated from the solution of the scalar wave equation

$$\nabla^2 \Psi + k^2 \Psi = 0 .$$

In spherical coordinates, the solutions of the scalar wave equation are of the form

$$\psi^{1}_{\substack{e\\o}\, mn}(\underline{r}) = \frac{\cos}{\sin}(m\,\phi)\; P_n^m(\cos\,\Theta)\; j_n(kr)$$

and

$$\psi^{3}_{\substack{e\\o}\, mn}(\underline{r}) = \frac{\cos}{\sin}(m\,\phi)\; P_n^m(\cos\,\Theta)\; h_n^{(1)}(kr)$$

where $n = 0, 1, 2, \ldots;\ m = 0, 1, \ldots, n;\ P_n^m(\cos\,\Theta)$ is the associated Legendre Polynomial; $j_n(kr)$ is the spherical Bessel function of order n and $h_n^{(1)}(kr)$ is the spherical Hankel function of the first kind of order n.[3] In this work Hankel functions of the first kind alone will be used. Hence we shall drop the superscript for convenience and write $h_n(kr)$ instead of $h_n^{(1)}(kr)$ throughout. The spherical Bessel functions are used inside the sphere including the origin, and the spherical Hankel functions outside the sphere (since they are regular at infinity and have a pole at the origin). The vectors

$$\underline{l}(\underline{r}) = \nabla\psi(\underline{r});\ \underline{m}(\underline{r}) = \nabla\times[\underline{r}\,\psi(\underline{r})]\ \text{and}\ \underline{n}(\underline{r}) = \frac{1}{k}\nabla\times\underline{m}(\underline{r})$$

satisfy the vector wave equation. Since the electric and magnetic fields are solenoidal (i.e., their divergence is zero) only the functions $\underline{m}(\underline{r})$ and $\underline{n}(\underline{r})$ are used. As shown in Stratton [1943], the incident wave can be expressed in terms of the spherical vector wave functions

$$\underline{m}^{1,3}_{\substack{e\\o}\, mn}(\underline{r}) = \nabla\times[\underline{r}\,\psi^{1,3}_{\substack{e\\o}\, mn}(\underline{r})] = [\nabla\psi^{1,3}_{\substack{e\\o}\, mn}(\underline{r})]\times\underline{r}$$

---

[3] Definitions and notations here follow those given by Morse and Feshbach [1953].

and

$$\underline{n}^{1,3}_{\substack{e\\o}mn}(\underline{r}) = \frac{1}{k}\nabla\times\underline{m}^{1,3}_{\substack{e\\o}mn}(\underline{r})$$

as follows:

$$\underline{E}^i(\underline{r}) = \hat{i}_x e^{ikz} = \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)}\left[\underline{m}^1_{o1n}(\underline{r},k) - i\underline{n}^1_{e1n}(\underline{r},k)\right] \quad (17)$$

where

$$\underline{m}^1_{o1n}(\underline{r},k) = \nabla\left[\sin\phi\, P^1_n(\cos\theta)\, j_n(kr)\right]\times\underline{r}$$

The scattered field outside a sphere centered at the origin is given by

$$\underline{E}^s(\underline{r}) = \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)}\left[a^s_n\,\underline{m}^3_{o1n}(\underline{r},k) - i\, b^s_n\,\underline{n}^3_{e1n}(\underline{r},k)\right], \quad r > a \quad (18)$$

and the field inside the sphere, the "transmitted" field, is given by

$$\underline{E}^t(\underline{r}) = \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)}\left[a^t_n\,\underline{m}^1_{o1n}(\underline{r},k_s) - i\, b^t_n\,\underline{n}^1_{e1n}(\underline{r},k_s)\right], \quad r < a \quad (19)$$

The coefficients a^s_n, b^s_n, a^t_n and b^t_n are determined from the boundary conditions which require continuity of the $\underline{E}$- and $\underline{H}$- fields across the surface of the sphere. For a sphere of radius a, propagation constant k_s, permeability μ_s and dielectric constant ϵ_s embedded in a medium of constants k, μ, and ϵ, these coefficients are given by

$$a_n^s = -\frac{\mu_s j_n(N_s\zeta)[\zeta j_n(\zeta)]' - \mu j_n(\zeta)[N_s\zeta j_n(N_s\zeta)]'}{\mu_s j_n(N_s\zeta)[\zeta h_n(\zeta)]' - \mu h_n(\zeta)[N_s\zeta j_n(N_s\zeta)]'} \tag{20a}$$

$$b_n^s = -\frac{\mu_s j_n(\zeta)[N_s\zeta j_n(N_s\zeta)]' - \mu N_s^2 j_n(N_s\zeta)[\zeta j_n(\zeta)]'}{\mu_s h_n(\zeta)[N_s\zeta j_n(N_s\zeta)]' - \mu N_s^2 j_n(N_s\zeta)[\zeta h_n(\zeta)]'} \tag{20b}$$

$$a_n^t = \mu_s \frac{h_n(\zeta)[\zeta j_n(\zeta)]' - j_n(\zeta)[\zeta h_n(\zeta)]'}{\mu h_n(\zeta)[N_s\zeta j_n(N_s\zeta)]' - \mu_s j_n(N_s\zeta)[\zeta h_n(\zeta)]'} \tag{20c}$$

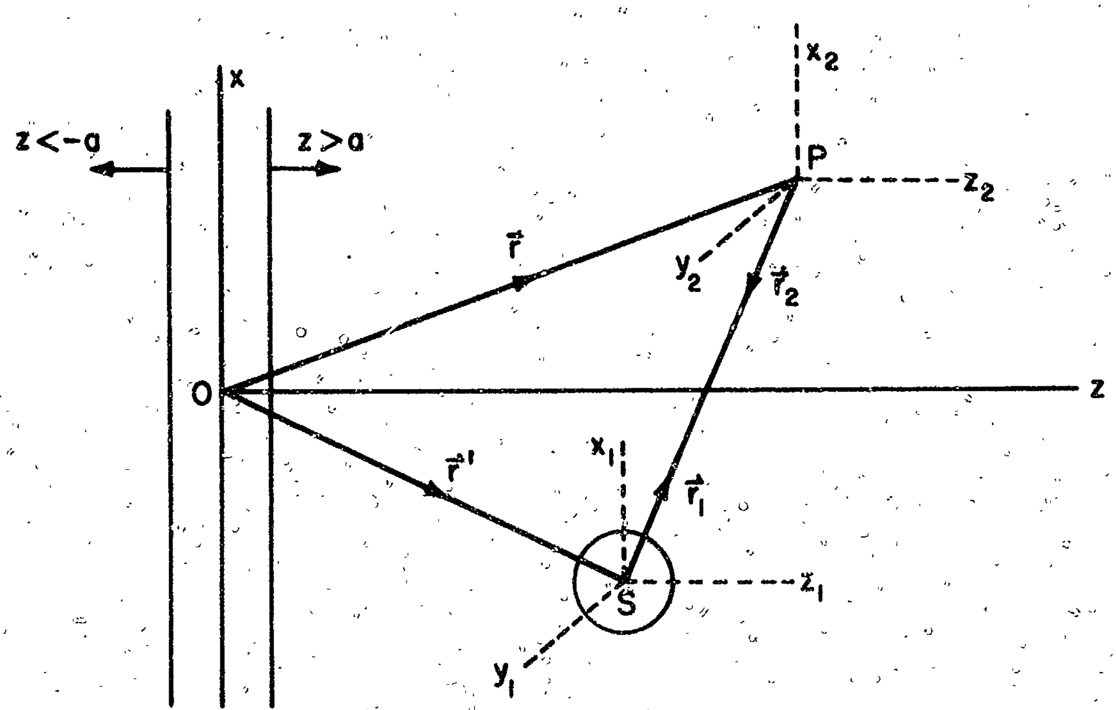
$$b_n^t = \mu_s N_s \frac{h_n(\zeta)[\zeta j_n(\zeta)]' - j_n(\zeta)[\zeta h_n(\zeta)]'}{\mu_s h_n(\zeta)[N_s\zeta j_n(N_s\zeta)]' - \mu N_s^2 j_n(N_s\zeta)[\zeta h_n(\zeta)]'} \tag{20d}$$

The notation used here is $ka = \zeta$ and $k_s = N_s k$ and the primes indicate differentiation with respect to the appropriate argument.

4.3 Integration of the Mie Series

The equations (17)-(19), giving the incident, scattered and transmitted fields, are true when the sphere is centered at the origin of coordinates. We are interested in the fields at an arbitrary point $\underline{r}$ when the sphere is centered at another arbitrary point $\underline{r}'$. Therefore, we use various coordinate axes as shown in Figure 2.

The original coordinate system (x,y,z) is centered at the origin 0. In addition, let the coordinate systems (x_1,y_1,z_1) with origin at S (the center of the scatterer) and (x_2,y_2,z_2) with origin at P (the point of observation) be rigid translations of the original coordinate system. The coordinates of a point with respect to the origin S will have subscript



$$\vec{r} = \vec{r}' + \vec{r}_1$$

$$= \vec{r}' - \vec{r}_2$$

Figure 2. The coordinate systems.

1 (e.g. x_1, y_1, z_1, or r_1, θ_1, ϕ_1) and those with respect to the origin P will have subscript 2. No subscripts are used when referring to the original system centered at 0.

We can now express the incident field as

$$\underline{E}^i(\underline{r}) = \hat{i}_x e^{ikz} = \hat{i}_x e^{ik(z'+z_1)}$$

$$= e^{ikz'} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} [\underline{m}^1_{oln}(\underline{r}_1, k) - i\, \underline{n}^1_{eln}(\underline{r}_1, k)]$$

Here $e^{ikz'}$ is a phase factor which depends upon the position of the scatterer. The scattered and transmitted fields are now given by

$$T(\underline{r},\underline{r}')\, \underline{E}^i(\underline{r}') = e^{ikz'} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} [a^s_n\, \underline{m}^3_{oln}(\underline{r}_1, k) - i\, b^s_n\, \underline{n}^3_{eln}(\underline{r}_1, k)]$$

$$T^I(\underline{r},\underline{r}')\, \underline{E}^i(\underline{r}') = e^{ikz'} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} [a^t_n\, \underline{m}^1_{oln}(\underline{r}_1, k_s) - i\, b^t_n\, \underline{n}^1_{eln}(\underline{r}_1, k_s)]$$

The coefficients a^s_n, b^s_n, a^t_n and b^t_n are given by equation (20). Substituting these results in equations (15) and (16) the average total field is given by

$$\langle \underline{E}(\underline{r}) \rangle = (1-v_s)\, \hat{i}_x e^{ikz}$$

$$+ (1-v_s)\rho_o \int_{\substack{|\underline{r}-\underline{r}'|>a, \\ z' \geq 0}} dv'\, e^{ikz'} [\sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \{ a^s_n\, \underline{m}^3_{oln}(\underline{r}_1) - i\, b^s_n\, \underline{n}^3_{eln}(\underline{r}_1) \}]$$

$$+ \rho_o \int_{\substack{|\underline{r}-\underline{r}'|<a \\ z' \geq 0}} dv'\, e^{ikz'} [\sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \{ a^t_n\, \underline{m}^1_{oln}(\underline{r}_1, k_s) - i\, b^t_n\, \underline{n}^1_{eln}(\underline{r}_1, k_s) \}] \qquad (21)$$

when $z > a$, and by

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} + \rho_o \int_{z' \geq 0} dv' \, e^{ikz'} \left[\sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \left\{ a_n^s \, \underline{m}^3_{oln}(\underline{r}_1) - i b_n^s \underline{n}^3_{eln}(\underline{r}_1) \right\} \right] \tag{22}$$

when $z < -a$. Here $v_s = \rho_o \frac{4}{3} \pi a^3 = \rho_o \int_{|\underline{r}-\underline{r}'|<a} dv'$ = fractional volume occupied by the scatterers.

4.31 Transformation of Coordinates

In order to carry out the integrations over $\underline{r}'$ involved in equations (21) and (22), we have to carry out translations of the vector wave function $\underline{m}(\underline{r}_1)$, $\underline{n}(\underline{r}_1)$ so as to express them as functions of $\underline{r}$ and $\underline{r}'$. Considerable work has been done on the translation of spherical vector wave functions. The resulting addition theorems are expressed in terms of triple infinite series involving very complicated coefficients (for instance, see Cruzan [1962]). We shall avoid the use of this procedure and, instead, use the following simple and elegant technique.

The variable of integration is the scatterer center S. The point of observation P is a fixed point in the integration. The restrictions $|\underline{r}-\underline{r}'| > a$ and $|\underline{r}-\underline{r}'| < a$ on the domain of integration merely restrict S to be outside or inside a sphere of radius a, centered at P. So we transform the various functions in the integrand to a coordinate system with origin at P. The relevant vector relations and the corresponding relations in terms of Cartesian and spherical polar coordinates are

$$\underline{r}' = \underline{r} + \underline{r}_2 ; \quad z' = z + z_2$$

$$\underline{r}_1 = -\underline{r}_2 ; \quad r_1 = r_2 , \quad \Theta_1 = \pi - \Theta_2 , \quad \phi_1 = \phi_2 - \pi$$

Using these relations it can easily be shown that

$$e^{ikz'} = e^{ikz} e^{ikz_2}$$

$$\underline{m}^{1,3}_{oln}(\underline{r}_1, k) = (-1)^n \underline{m}^{1,3}_{oln}(\underline{r}_2, k)$$

$$\underline{n}^{1,3}_{eln}(\underline{r}_1, k) = (-1)^{n+1} \underline{n}^{1,3}_{eln}(\underline{r}_2, k)$$

Substituting these relations in equations (21) and (22), we get

$$\langle \underline{E}(\underline{r}) \rangle = (1-v_s) \hat{i}_x e^{ikz}$$

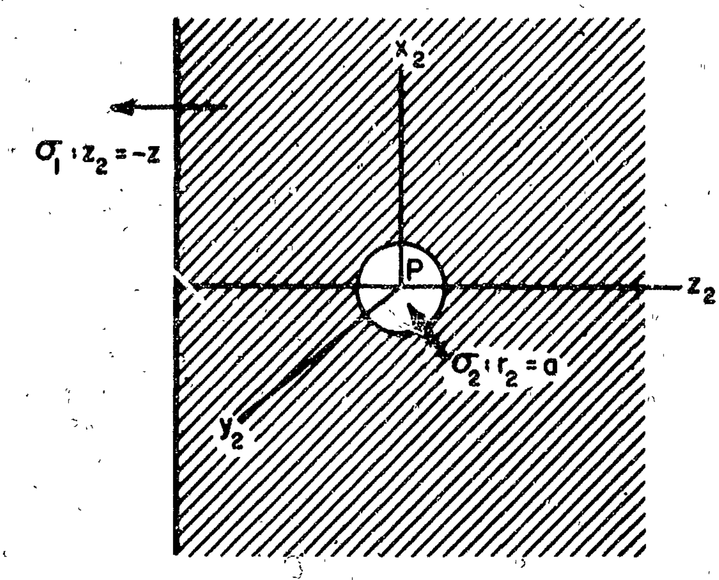
$$+ (1-v_s) \rho_o e^{ikz} \int_{\substack{r_2 > a \\ z_2 \geqq -z}} dv_2 \, e^{ikz_2} \Big[\sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} \left\{ a^s_n \underline{m}^3_{oln}(\underline{r}_2) + i \, b^s_n \underline{n}^3_{eln}(\underline{r}_2) \right\} \Big]$$

$$+ \rho_o e^{ikz} \int_{r_2 < a} dv_2 \, e^{ikz_2} \Big[\sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} \left\{ a^t_n \underline{m}^1_{oln}(\underline{r}_2, k_s) + i \, b^t_n \underline{n}^1_{eln}(\underline{r}_2, k_s) \right\} \Big] \tag{23}$$

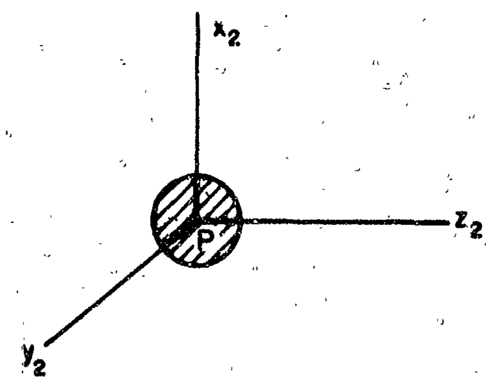
for the region $z > a$, and

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} + \rho_o e^{ikz} \int_{z_2 \geqq -z} dv_2 e^{ikz_2} \Big[\sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} \left\{ a^s_n \underline{m}^3_{oln}(\underline{r}_2) + i b^s_n \underline{n}^3_{eln}(\underline{r}_2) \right\} \Big] \tag{24}$$

for $z \leqq -a$. The regions of integration are shown in Figure 3.



(a) $r_2 > a$, $z_2 \geqq -z$; $z > a$



(b) $r_2 < a$

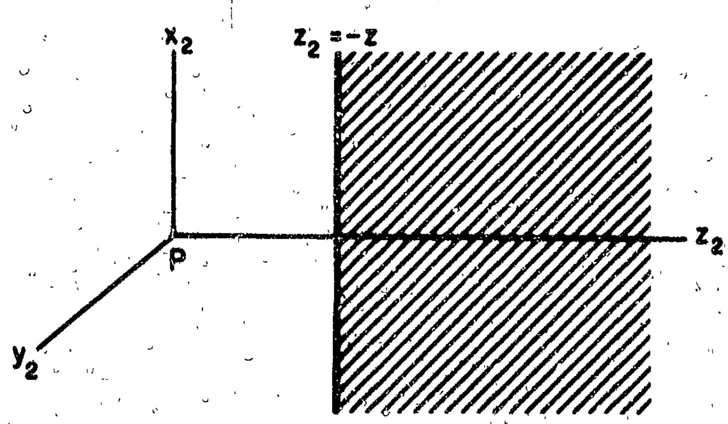


Figure 3. The domains of integration (shaded region).

4.32 Expansion of the Integrands

The infinite series in the integrands are uniformly convergent in the domains of integration. The integration and summation can, therefore, be interchanged. The expansions of the spherical vector wave functions in terms of the Cartesian components can be easily obtained from the defining equations. For the functions $\underline{m}^{1,3}_{oln}(\underline{r}_2)$ and $\underline{n}^{1,3}_{eln}(\underline{r}_2)$, these expansions are given by (see Morse and Feshbach [1953])

$$\underline{m}^{1,3}_{oln}(\underline{r}_2,k) = \hat{i}_x\left[\frac{n(n+1)}{2}P_n z_n + \frac{1}{2}P^2_n z_n \cos 2\phi_2\right] + \hat{i}_y\left[\frac{1}{2}P^2_n z_n \sin 2\phi_2\right] + \hat{i}_z\left[-P^1_n z_n \cos\phi_2\right] \tag{25}$$

and

$$\underline{n}^{1,3}_{eln}(\underline{r}_2,k) = \hat{i}_x\Bigg[\frac{n(n+1)}{(2n+1)^2}(z_{n-1}+z_{n+1})\left\{\frac{n(n+1)}{2}(P_{n-1}-P_{n+1}) - \frac{\cos 2\phi_2}{2}(P^2_{n-1}-P^2_{n+1})\right\}$$

$$+\frac{n(n+1)}{(2n+1)^2}\left(\sqrt{\frac{n+1}{n}}\,z_{n-1} - \sqrt{\frac{n}{n+1}}\,z_{n+1}\right)\left\{\frac{n(n+1)}{2}\left(\sqrt{\frac{n+1}{n}}\,P_{n-1} + \sqrt{\frac{n}{n+1}}\,P_{n+1}\right)\right.$$

$$\left.-\frac{\cos 2\phi_2}{2}\left(\sqrt{\frac{n+1}{n}}\,P^2_{n-1} + \sqrt{\frac{n}{n+1}}\,P^2_{n+1}\right)\right\}\Bigg]$$

$$+\hat{i}_y\Bigg[\frac{n(n+1)}{(2n+1)^2}(z_{n-1}+z_{n+1})\left\{-\frac{\sin 2\phi_2}{2}(P^2_{n-1}-P^2_{n+1})\right\}$$

$$+\frac{n(n+1)}{(2n+1)^2}\left(\sqrt{\frac{n+1}{n}}\,z_{n-1} - \sqrt{\frac{n}{n+1}}\,z_{n+1}\right)\left\{-\frac{\sin 2\phi_2}{2}\left(\sqrt{\frac{n+1}{n}}\,P^2_{n-1} + \sqrt{\frac{n}{n+1}}\,P^2_{n+1}\right)\right\}\Bigg]$$

$$+\hat{i}_z\Bigg[\frac{n(n+1)}{(2n+1)^2}(z_{n-1}+z_{n+1})\cos\phi_2\left\{(n+1)P^1_{n-1} + n\,P^1_{n+1}\right\}$$

$$+\frac{n(n+1)}{(2n+1)^2}\left(\sqrt{\frac{n+1}{n}}\,z_{n-1} - \sqrt{\frac{n}{n+1}}\,z_{n+1}\right)\cos\phi_2\left\{\sqrt{\frac{n+1}{n}}(n+1)P^1_{n-1} - \sqrt{\frac{n}{n+1}}\,n\,P^1_{n+1}\right\}\Bigg] \tag{26}$$

Here z_n stands for j_n when the superscript is 1 and for h_n when the superscript is 3. All associated Legendre polynomials have the argument $(\cos\theta_2)$ and z_n has the argument (kr_2).

The domains of integration exhibit a symmetry about the z_2-axis and since

$$\int_0^{2\pi} d\,\phi_2 \sin m\,\phi_2 = \int_0^{2\pi} d\,\phi_2 \cos m\,\phi_2 = 0\,, \qquad m = 1, 2, \ldots,$$

the only non-vanishing terms will be those which have integrands independent of ϕ_2. An examination of equations (25) and (26) shows that the $\hat{i}_y$ and $\hat{i}_z$ components will have zero contribution. In fact, in view of these considerations, equations (23) and (24) reduce to

$$\begin{aligned}
\langle \underset{\sim}{E}(\underset{\sim}{r})\rangle &= (1-v_s)\,\hat{i}_x\, e^{ikz} \\
&+ (1-v_s)\,\hat{i}_x e^{ikz} \rho_o \sum_{n=1}^{\infty} \frac{(-i)^n}{2} \int_{\substack{r_2 > a \\ z_2 \gtreqless -z}} dv_2\, e^{ikz_2} [\,(2n+1)\, a_n^s P_n(\cos\theta_2) h_n(kr_2) \\
&+ i\, b_n^s \left\{(n+1)\, P_{n-1}(\cos\theta_2) h_{n-1}(kr_2) - n\, P_{n+1}(\cos\theta_2)\, h_{n+1}(kr_2)\right\}] \\
&+ \hat{i}_x e^{ikz} \rho_o \sum_{n=1}^{\infty} \frac{(-i)^n}{2} \int_{r_2 < a} dv_2\, e^{ikz_2} [\,(2n+1)\, a_n^t\, P_n(\cos\theta_2)\, j_n(k_s r_2) \\
&+ i\, b_n^t \left\{(n+1)\, P_{n-1}(\cos\theta_2)\, j_{n-1}(k_s r_2) - n\, P_{n+1}(\cos\theta_2)\, j_{n+1}(k_s r_2)\right\}]
\end{aligned} \tag{27}$$

for the region $z > a$, and

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} + \hat{i}_x e^{ikz} \rho_o \sum_{n=1}^{\infty} \frac{(-i)^n}{2} \int_{z_2 \geq -z} dv_2 e^{ikz_2} [(2n+1) a_n^s P_n(\cos \theta_2) h_n(kr_2)$$

$$+ i b_n^s \{ (n+1) P_{n-1}(\cos \theta_2) h_{n-1}(kr_2) - n P_{n+1}(\cos \theta_2) h_{n+1}(kr_2) \}] \tag{28}$$

for the region $z < -a$.

4.33 Techniques of Integration

The integrands involved in the above equations are essentially of the form $e^{ikz_2} P_n(\cos \theta_2) h_n(kr_2)$ for the domains (a) and (c) and of the form $e^{ikz_2} P_n(\cos \theta_2) j_n(k_s r_2)$ for the domain (b) shown in Figure 3.

In the domain (a), the exclusion of the spherical volume $r_2 < a$ insures that there is no singularity. However, it also makes straightforward integration impossible. A technique has been developed to change the volume integral to a surface integral. As shown in Appendix I, we have

$$\int_V dv_2 \, e^{ikz_2} P_n(\cos \theta_2) \, h_n(kr_2) = \int_\sigma [P_n(\cos \theta_2) h_n(kr_2) \nabla \left\{ e^{ikz_2} \left(\frac{1}{4k^2} - \frac{iz_2}{2k}\right) \right\}$$

$$- e^{ikz_2} \left(\frac{1}{4k^2} - \frac{iz_2}{2k}\right) \nabla \left\{ P_n(\cos \theta_2) \, h_n(kr_2) \right\}] \cdot \underline{dS}$$

where the surface σ encloses the volume V and $\underline{dS}$ is the outward normal. In Figure (3a), the surface $\sigma = \sigma_1 + \sigma_2$ with the outward normals as shown. For σ_1 the outward normal is in the negative z_2-direction and, therefore, the gradient can be replaced by $(- \frac{\partial}{\partial z_2})$ and dS by $2\pi\rho_2 d\rho_2$ (using cylindrical coordinates). For σ_2, the outward normal is in the inward radial direction and we replace the gradient by $(-\frac{\partial}{\partial r_2})$ and dS by $2\pi a^2 \sin\Theta_2 d\Theta_2$. So we have

$$\int_{r_2>a,\, z_2 \geq -z} dv_2\, e^{ikz_2} P_n(\cos\Theta_2)\, h_n(kr_2) = I_{\sigma_1} + I_{\sigma_2} .$$

Now

$$I_{\sigma_1} = \int_0^\infty 2\pi\rho_2 d\rho_2 \left[P_n(\cos\Theta_2)\, h_n(kr_2) \frac{\partial}{\partial z_2} \left\{ e^{ikz_2} \left(\frac{iz_2}{2k} - \frac{1}{4k^2}\right) \right\} \right.$$

$$\left. - e^{ikz_2} \left(\frac{iz_2}{2k} - \frac{1}{4k^2}\right) \frac{\partial}{\partial z_2} \left\{ P_n(\cos\Theta_2)\, h_n(kr_2) \right\} \right]_{z_2=-z}$$

The cylindrical coordinates and the spherical coordinates have the following relationships

$$\rho_2^2 + z_2^2 = r_2^2 ; \quad z_2 = r_2 \cos\Theta_2$$

and

$$P_n(\cos\Theta_2)\, h_n(kr_2) = (-i)^n P_n\left(\frac{1}{ik}\frac{\partial}{\partial z_2}\right) \frac{e^{ikr_2}}{ikr_2} \qquad (29)$$

A proof of equation (29) is given in Appendix II. Remembering that in the domain (a) $z > a$ and, therefore, a positive number, we have

$$I_{\sigma_1} = 2\pi(-i)^n[e^{ikz_2}(\frac{i}{4k} - \frac{z_2}{2})\, P_n(\frac{1}{ik}\frac{\partial}{\partial z_2}) \int_0^\infty \rho_2 d\rho_2 \frac{e^{ik\sqrt{z_2^2+\rho_2^2}}}{ik\sqrt{z_2^2+\rho_2^2}}\,]_{z_2=-z}$$

$$- 2\pi(-i)^n[e^{ikz_2}(\frac{iz_2}{2k} - \frac{1}{4k^2})\frac{\partial}{\partial z_2}\left\{P_n(\frac{1}{ik}\frac{\partial}{\partial z_2}) \int_0^\infty \rho_2 d\rho_2 \frac{e^{ik\sqrt{z_2^2+\rho_2^2}}}{ik\sqrt{z_2^2+\rho_2^2}}\right\}]_{z_2=-z}$$

$$= 2\pi(-i)^n[e^{ikz_2}(\frac{i}{4k} - \frac{z_2}{2})\, P_n(\frac{1}{ik}\frac{\partial}{\partial z_2}) \frac{e^{ik|z_2|}}{k^2}$$

$$- e^{ikz_2}(\frac{iz_2}{2k} - \frac{1}{4k^2})\frac{\partial}{\partial z_2}\left\{P_n(\frac{1}{ik}\frac{\partial}{\partial z_2}) \frac{e^{ik|z_2|}}{k^2}\right\}]_{z_2=-z}$$

$$= \frac{2\pi i^n}{k^2}\, z$$

where we have used the relations

$$P_n(-x) = (-1)^n P_n(x)$$

and

$$P_n(\frac{1}{ik}\frac{\partial}{\partial z_2})\, e^{\pm ikz_2} = P_n(\pm 1)\, e^{\pm ikz_2} = (\pm 1)^n e^{\pm ikz_2}$$

Similarly we have

$$I_{\sigma_2} = 2\pi a^2 \int_0^{\pi} \sin\Theta_2\, d\Theta_2 [P_n(\cos\Theta_2)\, h_n(kr_2) \frac{\partial}{\partial r_2} \left\{ e^{ikz_2} \left(\frac{iz_2}{2k} - \frac{1}{4k^2}\right) \right\}$$

$$- e^{ikz_2} \left(\frac{iz_2}{2k} - \frac{1}{4k^2}\right) \frac{\partial}{\partial r_2} \left\{ P_n(\cos\Theta_2)\, h_n(kr_2) \right\}]_{r_2 = a}$$

This integral is straightforward and after some computations making use of the recurrence relations satisfied by Legendre Polynomials and spherical Hankel functions and the well known relation (see Morse and Feshbach [1953])

$$\int_0^{\pi} e^{ika\cos\Theta} P_n(\cos\Theta) \sin\Theta\, d\Theta = 2i^n j_n(ka)$$

we get

$$I_{\sigma_2} = \frac{\pi a^2 i^n}{k} [h_n(\zeta) \left\{ 2\zeta\, j_n''(\zeta) + j_n'(\zeta) \right\} - h_n'(\zeta) \left\{ 2\zeta j_n'(\zeta) - j_n(\zeta) \right\}]$$

where the primes indicate differentiation with respect to the argument. We can, therefore, write

$$\int_{r_2 > a,\, z_2 \geqq -z} dv_2\; e^{ikz_2} P_n(\cos\Theta_2)\, h_n(kr_2) = \frac{\pi i^n}{k^3} [2kz + a_n] , \quad z > a \quad (30)$$

where we define

$$a_n = \zeta^2 [h_n(\zeta) j_n'(\zeta) + h_n'(\zeta) j_n(\zeta) + 2\zeta \left\{ h_n(\zeta) j_n''(\zeta) - h_n'(\zeta) j_n'(\zeta) \right\}] \quad (31)$$

In the domain (b) of Figure 3, the integration is easily carried out as below

$$\int_{r_2 < a} dv_2\, e^{ikz_2} P_n(\cos\theta_2)\, j_n(k_s r_2)$$

$$= 2\pi \int_0^a r_2^2\, j_n(k_s r_2)\, dr_2 \int_0^\pi e^{ikr_2 \cos\theta_2} P_n(\cos\theta_2) \sin\theta_2\, d\theta_2$$

$$= \frac{4\pi i^n a^2}{k^2 - k_s^2} [k_s j_n(ka) j_{n-1}(k_s a) - k\, j_{n-1}(ka)\, j_n(k_s a)]$$

$$= \frac{\pi i^n}{k^3}\, \beta_n \tag{32}$$

where we define

$$\beta_n = \frac{4\zeta^2}{1-N_s^2} [N_s j_n(\zeta)\, j_{n-1}(N_s\zeta) - j_{n-1}(\zeta)\, j_n(N_s\zeta)] \tag{33}$$

In the domain (c) of Figure 3, $z < -a$ and we have

$$\int_{z_2 \geqq -z} dv_2\, e^{ikz_2} P_n(\cos\theta_2) h_n(kr_2) = 2\pi(-i)^n \int_{-z}^{\infty} dz_2\, e^{ikz_2} P_n\left(\frac{1}{ik}\frac{\partial}{\partial z_2}\right) \int_0^\infty \rho_2 d\rho_2 \frac{e^{ikr_2}}{ikr_2}$$

$$= \frac{\pi(-i)^{n-1}}{k^3}\, e^{-i2kz} \tag{34}$$

We now have all the integrations of equations (27) and (28) and are in a position to write down the final results

4.4 Total Field in the Born Approximation

For a point of observation in the region $z > a$, the average total field is given by

$$\langle \underline{E}(\underline{r}) \rangle = (1-v_s)\,\hat{i}_x\, e^{ikz}$$

$$+ (1-v_s)\,\rho_o\,\hat{i}_x\, e^{ikz} \left[\frac{\pi z}{k^2} \sum_{n=1}^{\infty} (2n+1)\,(a_n^s + b_n^s) \right.$$

$$\left. + \frac{\pi}{2k^3} \sum_{n=1}^{\infty} \left\{ (2n+1)\, a_n^s\, \alpha_n + (n+1)\, b_n^s\, \alpha_{n-1} + n\, b_n^s\, \alpha_{n+1} \right\} \right]$$

$$+ \rho_o \hat{i}_x\, e^{ikz} \left[\frac{\pi}{2k^3} \sum_{n=1}^{\infty} \left\{ (2n+1)\, a_n^t\, \beta_n + (n+1)\, b_n^t\, \beta_{n-1} + n\, b_n^t\, \beta_{n+1} \right\} \right] \qquad (35)$$

For a point of observation in the region $z < -a$, the field is given by

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x\, e^{ikz} + \hat{i}_x\, e^{-ikz} \left[\frac{\rho_o \pi i}{2k^3} \sum_{n=1}^{\infty} (-1)^n\, (2n+1)\, (a_n^s - b_n^s) \right] \qquad (36)$$

The most important result is that the polarization of the average total field is the same as that of the incident field. Another result is seen from equation (36) which is of the form

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x\, e^{ikz} + \hat{i}_x\, E_1^r\, e^{-ikz} , \qquad z < -a$$

This shows that the right half space containing the scatterers acts like a modified medium which reflects part of the incident field. The "reflection coefficient" E_1^r (the subscript 1 indicates the first order theory) is determined by the size and density of scatterers and the wavelength. The behavior of the right half space as a modified homogeneous medium is also seen from equation (35) which can be written as

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x E_1^t e^{ikz} (1 + i\delta kz) , \qquad z > a$$

If δ is small (as it will be for situations in which the Born approximation is reasonably good), we can write

$$e^{ikz}(1 + i\delta kz) \approx e^{iN_B kz}$$

where $N_B \doteq 1 + \delta$

Thus the modified medium has a refractive index N_B and a "transmission coefficient" E_1^t. Within this medium the incident field is extinguished as would be expected.

5. Multiple Scattering by Spherical Scatterers

It has been pointed out earlier that instead of solving the integral equation (13) for the exciting field, we can obtain the average total field to various degrees of accuracy by successive iteration. The first iteration, which is the well known Born approximation, has been considered in Chapter 4 and expressions for the average total field have been obtained. For the second and higher iterations, the complexity of the integrals involved increases very rapidly. This is because we are considering the very general case of vector waves and scatterers of arbitrary size. In this chapter we shall consider an alternate approach and shall study the effects of multiple scattering through the exciting field as governed by equation (13).

5.1 Evaluation of Exciting Field Using Two-Exterior Formalism

Most of the earlier work on multiple scattering by small scatterers has shown that the distribution of scatterers can be replaced by a modified homogeneous medium. Thus Foldy [1945] has obtained an expression for the refractive index of such a modified medium for the case of isotropic point scatterers. A similar result for anisotropic point scatterers has been obtained by Waterman and Truell [1961] for scalar waves. The single-scattering approach of Chapter 4 gives the refractive index of the modified medium when vector waves are considered and no restriction is placed on the size of the scatterers. On the basis of these results, we shall assume that the exciting field can be represented by a collection of uniform plane wave modes when multiple scattering effects are taken into account. The multiplicity of these modes arises due to spatial dispersion effects. From the geometry of the problem and the results of Born approximation, it is clear that these plane waves will all travel in the positive z-direction like the incident wave and will all be linearly polarized with a polarization similar to that of the incident wave. Therefore, let the exciting field be given by

$$<\underline{E}^E(\underline{r}:\underline{r})> = \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z}$$

where all k_ℓ's are assumed to be distinct i.e., $k_\ell \neq k_{\ell'}$ for $\ell \neq \ell'$. Substituting this in equation (13) we get

$$\sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z} = \hat{i}_x e^{ikz} + \rho_o \int_{\substack{|\underline{r}-\underline{r}'|>2a \\ z' \geq 0}} dv' \, [T(\underline{r},\underline{r}') \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z'}]$$

In order to carry out the integration, we need to know $[T(\underline{r},\underline{r}') \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z'}]$ which is the scattered field at $\underline{r}$ from a scatterer at $\underline{r}'$ excited by the collection of plane waves of the type $\hat{i}_x E_\ell e^{ik_\ell z}$. Each of these plane waves gives rise to a scattered field which travels in the medium of constant k and a transmitted field inside the scatterer where the propagation constant is k_s. Thus, using the coordinate systems of Figure 2, we can write the incident, scattered, and transmitted fields of each plane wave mode as follows

$$i_x E_\ell e^{ik_\ell z} = E_\ell e^{ik_\ell z'} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} [\underline{m}^1_{oln}(\underline{r}_1, k_\ell) - i\, \underline{n}^1_{eln}(\underline{r}_1, k_\ell)] \tag{37a}$$

$$T(\underline{r}, \underline{r}')\hat{i}_x E_\ell e^{ik_\ell z'} = E_\ell e^{ik_\ell z'} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} [A^s_{\ell n} \underline{m}^3_{oln}(\underline{r}_1, k) - i\, B^s_{\ell n} \underline{n}^3_{eln}(\underline{r}_1, k)] \tag{37b}$$

$$T^I(\underline{r}, \underline{r}')\hat{i}_x E_\ell e^{ik_\ell z'} = E_\ell e^{ik_\ell z'} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} [A^t_{\ell n} \underline{m}^1_{Oln}(\underline{r}_1, k_s) - i\, B^t_{\ell n} \underline{n}^1_{eln}(\underline{r}_1, k_s)] \tag{37c}$$

This is the so-called "two-exterior" formalism of Twersky [1962a] indicating that the incident and scattered fields travel in two different media. The coefficients $A^s_{\ell n}$, $B^s_{\ell n}$, $A^t_{\ell n}$ and $B^t_{\ell n}$ for the plane wave mode ℓ are obtained from boundary conditions satisfied by the various fields on the surface of the isolated "schizoid" sphere. This problem is dealt with in Appendix III and the various coefficients are evaluated there. Using equation (37b) we see that the integral to be evaluated is

$$\int\limits_{\substack{|\underline{r}-\underline{r}'| \\ z' \geq 0}} dv'\, E_\ell e^{ik_\ell z'} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} [A^s_{\ell n} \underline{m}^3_{oln}(\underline{r}_1, k) - i\, B^s_{\ell n} \underline{n}^3_{eln}(\underline{r}_1, k)]$$

This is very similar to the one that was treated in Chapter 4. We use the transformation

$$\underline{r}_2 = -\underline{r}_1 = \underline{r}' - \underline{r}$$

and thereby refer the integrand to a coordinate system with origin at $\underline{r}$.

The domain of integration is similar to that shown in Figure 3(a) except that a spherical volume of radius 2a, rather than a, is excluded. The symmetry about the z_2-axis shows that the ϕ_2-integration reduces all but a few $\hat{i}_x$-component terms to zero. The above integral, therefore, reduces to

$$\hat{i}_x E_\ell e^{ik_\ell z} \sum_{n=1}^{\infty} \frac{(-i)^n}{2} \int_{\substack{r_2 \geq 2a \\ z_2 \geq -z}} dv_2\, e^{ik_\ell z_2} \left[(2n+1) A^s_{\ell n} P_n(\cos \Theta_2)\, h_n(kr_2) \right.$$

$$\left. + i\, B^s_{\ell n} \left\{ (n+1) P_{n-1}(\cos \Theta_2)\, h_{n-1}(kr_2) - n\, P_{n+1}(\cos \Theta_2)\, h_{n+1}(kr_2) \right\} \right]$$

The volume integral can be changed into a surface integral by using the relations

$$(\nabla_2^2 + k_\ell^2)\, e^{ik_\ell z} = 0$$

$$(\nabla_2^2 + k^2)\, P_n(\cos \Theta_2)\, h_n(kr_2) = 0 \qquad r_2 \neq 0$$

and Green's Theorem. Thus for $k_\ell \neq k$, we have

$$\int_V dv_2\, e^{ik_\ell z_2} P_n(\cos \Theta_2)\, h_n(kr_2) = \frac{1}{k_\ell^2 - k^2} \int_\sigma \left[e^{ik_\ell z_2} \nabla_2 \left\{ P_n(\cos \Theta_2)\, h_n(kr_2) \right\} \right.$$

$$\left. - P_n(\cos \Theta_2)\, h_n(kr_2)\, \nabla_2\, e^{ik_\ell z_2} \right] \cdot \underline{dS}$$

The surface σ is made up of $\sigma_1(z_2 = -z,\ z > 0)$ and $\sigma_2(r_2 = 2a)$. The integrations can be carried out easily along the lines of those in Chapter 4 and we get the following result

$$\int_{\substack{r_2 \geq 2a \\ z_2 \geq -z}} dv_2\, e^{ik_\ell z_2} P_n(\cos\Theta_2)\, h_n(kr_2) = \frac{2\pi i}{k^3(N_\ell - 1)} i^n e^{i(k-k_\ell)z} + \frac{4\pi i^n}{k^3(N_\ell^2 - 1)} \gamma_{\ell n} \tag{38}$$

where $N_\ell = k_\ell/k$ and

$$\gamma_{\ell n} = (2\zeta)^2 [N_\ell j_{n-1}(2N_\ell\zeta)\, h_n(2\zeta) - j_n(2N_\ell\zeta)\, h_{n-1}(2\zeta)] \tag{39}$$

Using this result in the exciting field equation we get

$$\sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z} = \hat{i}_x e^{ikz} + \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ikz} \frac{\pi i \rho_o}{k^3(N_\ell - 1)} \left[\sum_{n=1}^{\infty} (2n+1)(A_{\ell n}^s + B_{\ell n}^s) \right]$$

$$+ \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z} \frac{2\pi \rho_o}{k^3(N_\ell^2 - 1)} \left[\sum_{n=1}^{\infty} \left\{ (2n+1) A_\ell^s \gamma_{\ell n} + (n+1) B_{\ell n}^s \gamma_{\ell, n-1} + n B_{\ell n}^s \gamma_{\ell, n+1} \right\} \right] \tag{40}$$

Since this equation is true for all values of z in the right-half space, we can equate the coefficients of $e^{ik_\ell z}$, for all ℓ, and of e^{ikz} and get the following equations

$$\sum_{n=1}^{\infty} [(2n+1) A_{\ell n}^s \gamma_{\ell n} + (n+1) B_{\ell n}^s \gamma_{\ell, n-1} + n B_{\ell n}^s \gamma_{\ell, n+1}] = \frac{2\zeta^3}{3v_s} (N_\ell^2 - 1) \tag{41}$$

$$\sum_{\ell=1}^{\infty} \rho_o E_\ell \frac{3iv_s}{4\zeta^3 (N_\ell - 1)} \left[\sum_{n=1}^{\infty} (2n+1)(A_{\ell n}^s + B_{\ell n}^s) \right] + 1 = 0 \tag{42}$$

Equation (41) is the dispersion relation governing the refractive index of the modified medium. Its roots are the different modes which the medium can sustain. By itself, equation (41) is insufficient for getting the refractive index N_ℓ. This is because, as is readily seen from the expressions for $A^s_{\ell n}$, $B^s_{\ell n}$ in Appendix III, this equation also involves the permeability μ_ℓ corresponding to the ℓ-mode. However, another equation involving the same two constants N_ℓ and μ_ℓ can be easily derived. From the geometry of the problem it is clear that the medium will behave in the same way if the incident wave is polarized in the y-direction. We can, therefore, start by taking the <u>H</u>-field in the x-direction and carry out the entire analysis in a similar way. The "two-exterior" formalism will give the scattered field in terms of the coefficients $C^s_{\ell n}$ and $D^s_{\ell n}$ (as discussed in Appendix III) and we shall get another equation similar to (41) as below

$$\sum_{n=1}^{\infty} \left[(2n+1)\, C^s_{\ell n}\, \gamma_{\ell n} + (n+1)\, D^s_{\ell n}\, \gamma_{\ell, n-1} + n D^s_{\ell n}\, \gamma_{\ell, n+1} \right] = \frac{2\zeta^3}{3v_s} (N_\ell^2 - 1) \tag{43}$$

Between equations (41) and (43) we can get a transcendental equation in which the only unknown is N_ℓ. The different modes will be governed by this equation.

5.2 Evaluation of the Average Total Field

The average total field can be derived in a straightforward manner using the plane wave representation of the exciting field. Equations (14) and (5) can be written as follows

$$\langle \underline{E}(\underline{r}) \rangle \doteq (1-v_s)\,\hat{i}_x e^{ikz} + (1-v_s)\,\rho_o \int_{|\underline{r}-\underline{r}'|>a} dv' \left[T(\underline{r},\underline{r}') \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z'} \right]$$

$$+ \rho_o \int_{|r-r'|<a} dv' \left[T^I(\underline{r},\underline{r}') \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z'} \right]$$

when $z > a$, and

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} + \rho_o \int_{z' \geqq 0} dv' \left[T(\underline{r},\underline{r}') \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell e^{ik_\ell z'} \right]$$

when $z < -a$.

The "two-exterior" formalism of the last section gives the scattered and transmitted fields as expressed by equations 37(b) and 37(c). Using a coordinate system centered at P, the point of observation, the above equations reduce to

$$\langle \underline{E}(\underline{r}) \rangle = (1 - v_s)\,\hat{i}_x e^{ikz}$$

$$+ \sum_{\ell=1}^{\infty} (1-v_s)\rho_o E_\ell e^{ik_\ell z} \int_{\substack{r_2 \geqq a \\ z_2 \geqq -z}} dv_2\, e^{ik_\ell z_2} \Big[\sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} \Big\{ A^s_{\ell n} \underline{m}^3_{oln}(\underline{r}_2, k)$$

$$+ i\, B^s_{\ell n}\, \underline{n}^3_{eln}(\underline{r}_2, k) \Big\} \Big]$$

$$+ \sum_{\ell=1}^{\infty} \rho_o E_\ell e^{ik_\ell z} \int_{r_2 < a} dv_2\, e^{ik_\ell z_2} \Big[\sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} \Big\{ A^t_{\ell n} \underline{m}^1_{oln}(\underline{r}_2, k_s)$$

$$+ i\, B^t_{\ell n}\, \underline{n}^1_{eln}(\underline{r}_2, k_s) \Big\} \Big] \qquad (44)$$

when $z > a$, and

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} + \sum_{\ell=1}^{\infty} \rho_o E_\ell e^{ik_\ell z} \int_{z_2 \geq -z} dv_2 e^{ik_\ell z_2} \left[\sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} \left\{ A^s_{\ell n} \underline{m}^3_{oln}(\underline{r}_2, k) + i B^s_{\ell n} \underline{n}^3_{eln}(\underline{r}_2, k) \right\} \right] \tag{45}$$

when $z < -a$.

The domains of integration are those shown in Figure 3. As discussed earlier, the axial symmetry of these domains and the nature of the spherical vector wave functions reduce most of the terms to zero. The remaining terms involve integrals of the type

$$\int dv_2 \; e^{ik_\ell z_2} \; P_n(\cos \Theta_2) \; h_n(kr_2)$$

and
$$\int dv_2 \; e^{ik_\ell z_2} \; P_n(\cos \Theta_2) \; j_n(k_s r_2)$$

Let us consider the field in the region $z > a$ first. In equation (44), the second term has the domain of integration shown in Figure 3(a). Using equation (38) we get

$$\int_{\substack{r_2 \geq a \\ z_2 \geq -z}} dv_2 e^{ik_\ell z_2} P_n(\cos \Theta_2) h_n(kr_2) = \frac{2\pi i}{k^3 (N_\ell - 1)} i^n e^{i(k-k_\ell)z} + \frac{4\pi}{k^3 (N_\ell^2 - 1)} i^n \delta_{\ell n} \tag{46}$$

where $\delta_{\ell n}$ is defined by

$$\delta_{\ell n} = \zeta^2 [N_\ell j_{n-1}(N_\ell \zeta) \; h_n(\zeta) - j_n(N_\ell \zeta) \; h_{n-1}(\zeta)] \tag{47}$$

The integrals appearing in the third term of equation (44) can be evaluated as follows

$$\int_{r_2 < a} dv_2 \, e^{ik_\ell z_2} P_n(\cos\theta_2)\, j_n(k_s r_2)$$

$$= 2\pi \int_0^a dr_2 \, r_2^2 \, j_n(k_s r_2) \int_0^\pi d\theta_2 \sin\theta_2 \, e^{ik_\ell r_2 \cos\theta_2} P_n(\cos\theta_2)$$

$$= \frac{4\pi i^n a^2}{k_\ell^2 - k_s^2} [k_s \, j_n(k_\ell a)\, j_{n-1}(k_s a) - k_\ell \, j_{n-1}(k_\ell a)\, j_n(k_s a)]$$

$$= \frac{4\pi}{k^3 (N_\ell^2 - N_s^2)} \; i^n \epsilon_{\ell n} \tag{48}$$

where $\epsilon_{\ell n}$ is defined by

$$\epsilon_{\ell n} = \zeta^2 [N_s \, j_n(N_\ell \zeta)\, j_{n-1}(N_s \zeta) - N_\ell \, j_{n-1}(N_\ell \zeta)\, j_n(N_s \zeta)] \tag{49}$$

Combining these terms together, the average total field in the region $z > a$ is given by

$$\langle \underline{E}(\underline{r})\rangle = (1-v_s)\,\hat{i}_x\, e^{ikz}$$

$$+ \sum_{\ell=1}^{\infty} (1-v_s)\,\hat{i}_x E_\ell \rho_o \frac{\pi i}{k^3(N_\ell-1)}\, e^{ikz}\Big[\sum_{n=1}^{\infty} (2n+1)\,(A^s_{\ell n} + B^s_{\ell n})\Big]$$

$$+ \sum_{\ell=1}^{\infty} (1-v_s)\,\hat{i}_x E_\ell \rho_o \frac{2\pi}{k^3(N_\ell^2-1)}\, e^{ik_\ell z}\Big[\sum_{n=1}^{\infty}\big\{(2n+1)A^s_{\ell n}\delta_{\ell n} + (n+1)\,B^s_{\ell n}\delta_{\ell,n-1} + n\,B^s_{\ell n}\,\delta_{\ell,n+1}\big\}\Big]$$

$$+ \sum_{\ell=1}^{\infty} \hat{i}_x E_\ell \rho_o \frac{2\pi}{k^3(N_\ell^2-N_s^2)}\, e^{ik_\ell z}\Big[\sum_{n=1}^{\infty}\big\{(2n+1)A^t_{\ell n}\,\epsilon_{\ell n} + (n+1)B^t_{\ell n}\,\epsilon_{\ell,n-1} + n\,B^t_{\ell n}\,\epsilon_{\ell,n+1}\big\}\Big]$$

By virtue of equation (42), the first two terms of this equation add up to zero. The equation, therefore, reduces to the form

$$\langle \underline{E}(\underline{r})\rangle = \sum_{\ell=1}^{\infty} \hat{i}_x\, E^t_\ell\, e^{ik_\ell z} \quad , \quad z > a \tag{50}$$

where the transmission coefficients are given by

$$E^t_\ell = \frac{3v_s(1-v_s)}{2\zeta^3(N_\ell^2-1)}\, E_\ell \sum_{n=1}^{\infty}\big[(2n+1)A^s_{\ell n}\,\delta_{\ell n} + (n+1)B^s_{\ell n}\,\delta_{\ell,n-1} + n\,B^s_{\ell n}\,\delta_{\ell,n+1}\big]$$

$$+ \frac{3v_s}{2\zeta^3(N_\ell^2-N_s^2)}\, E_\ell \sum_{n=1}^{\infty}\big[(2n+1)A^t_{\ell n}\,\epsilon_{\ell n} + (n+1)B^t_{\ell n}\,\epsilon_{\ell,n-1} + nB^t_{\ell n}\,\epsilon_{\ell,n+1}\big] \tag{51}$$

Thus we see that the average total field propagates in the medium containing the scatterers as a collection of plane wave modes. The propagation constants of the various modes are determined by equations (41) and (43). The extinction theorem is verified since there is no e^{ikz} component in the field. It is seen that all modes are linearly polarized. It will be shown in the next chapter that when the spheres are very small compared to the wavelength, only one mode propagates and the propagation constant agrees with that derived by other authors.

Turning now to the average total field in the left half space, a typical integral in equation (45) has the domain of Figure 3(c) and can be evaluated as below

$$\int_{z_2 \geq -z} dv_2 \, e^{ik_\ell z_2} P_n(\cos \theta_2) \, h_n(kr_2)$$

$$= 2\pi(-i)^n \int_{-z}^{\infty} dz_2 \, e^{ik_\ell z_2} P_n\left(\frac{1}{ik} \frac{\partial}{\partial z_2}\right) \int_0^{\infty} \rho_2 \, d\rho_2 \, \frac{e^{ikr_2}}{ikr_2}$$

$$= 2\pi(-i)^n \int_{-z}^{\infty} dz_2 \, e^{ik_\ell z_2} \frac{e^{ikz_2}}{k^2}$$

$$= \frac{2\pi}{k^2(k_\ell+k)} (-i)^{n-1} e^{-i(k_\ell+k)z} \qquad (52)$$

Using this equation, (45) can be reduced to the form

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} + \sum_{\ell=1}^{\infty} \hat{i}_x E_{\ell o} \frac{\pi i}{k^3(N_\ell+1)} e^{-ikz} \left[\sum_{n=1}^{\infty} (-1)^n (2n+1)(A^s_{\ell n} - B^s_{\ell n}) \right]$$

This can be written in the form

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} + E^r \hat{i}_x e^{-ikz}, \quad z < -a \tag{53}$$

where the reflection coefficient is defined by

$$E^r = \sum_{\ell=1}^{\infty} \frac{3v_s i}{4\zeta^3 (N_\ell + 1)} E_\ell \sum_{n=1}^{\infty} (-1)^n (2n+1) (A_{\ell n}^s - B_{\ell n}^s) \tag{54}$$

Thus on the left of the scattering region, the total field is the sum of the incident field and a reflected field. The reflection coefficient is determined by the properties of the scatterers.

This treatment has given a fairly good picture of multiple scattering of electromagnetic waves by a random distribution of spheres of arbitrary size and material. It is by no means complete. There is not enough information to determine uniquely the amplitudes E_ℓ of the plane wave modes making up the exciting field. Because of the complexity of integrals, the treatment has excluded the infinite slab region $-a < z < a$ from the analysis. However, sufficient information has been obtained to determine the refractive index of the modified medium.

6. Scattering by Special Types of Spheres

In the last two chapters we have considered the behavior of the statistical expectation of the electric field in weakly random as well as strongly random media. The treatment was quite general and no restrictions were placed on either the size of the spherical scatterers or their electromagnetic properties. It is worthwhile to consider a few special cases and study the properties of the medium when certain constraints are placed on the scatterers. We shall consider propagation of low frequency waves for which the wavelength is much larger than the radius of the spheres. In this case the parameter $\zeta(=ka)$ is very small compared to unity and asymptotic expressions for spherical Bessel and Hankel functions for small argument can be used. We shall also consider the case of spheres of very large conductivity. In this case there are no fields interior to the spheres and the Mie series coefficients are considerably simplified.

6.1 Single Scattering Behavior

We have seen in Chapter 4 that the average total field propagates in the medium with an amplitude and phase velocity different from that of the incident wave but with the same polarization. The refractive index of the medium is given by an expression which is quite involved in the most general case. We shall consider two special cases.

6.11 Sphere Size Small Compared to Wavelength

For spheres small compared to wavelength (that is, at low frequencies) ζ is very small and the infinite series converges very fast. Asymptotic expansions of spherical Bessel and Neumann functions for small argument are of the form (see, for example, Gumprecht and Sliepcevich [1951])

$$j_n(\zeta) = \frac{2^n\, n!}{(2n+1)!}\, \zeta^n \left[1 - \frac{\zeta^2}{2(2n+3)} + \ldots\right]$$

$$n_n(\zeta) = -\frac{(2n)!}{2^n\, n!}\, \frac{1}{\zeta^{n+1}} \left[1 + \frac{\zeta^2}{2(2n-1)} + \ldots\right]$$

The Hankel function is, of course, defined by $h_n(\zeta) = j_n(\zeta) + i\, n_n(\zeta)$. If these expressions are used and terms of order higher than ζ^3 and $(N_s\zeta)^3$ are neglected, the refractive index of the weakly random medium is given by

$$N_B = 1 + \frac{\frac{3}{2} v_s \left[\frac{\mu_s-\mu}{\mu_s+2\mu} - \frac{\mu_s-\mu N_s^2}{2\mu_s+\mu N_s^2}\right]}{1 + \frac{v_s}{4}\left[\frac{\mu_s-\mu}{\mu_s+2\mu} - \frac{4}{5}\,\frac{\mu_s-\mu N_s^2}{2\mu_s+\mu N_s^2}\right] - \frac{3v_s}{1-v_s}\,\frac{\mu_s}{(2\mu_s-\mu N_s^2)}} \qquad (55)$$

If the permeability of the spheres is very nearly equal to that of the surrounding medium, the above expression simplifies to

$$N_B = 1 + \frac{\frac{3}{2} v_s \left[\frac{N_s^2 - 1}{N_s^2 + 2}\right]}{1 + \frac{v_s}{5}\left[\frac{N_s^2 - 1}{N_s^2 + 2}\right] - \frac{3v_s}{1-v_s}\,\frac{1}{(2-N_s^2)}}, \quad \mu_s \approx \mu \qquad (56)$$

If, in addition, we consider sparse concentration, the fractional volume occupied by the spheres, v_s, is very small. To the first power of v_s, then, we have

$$N_B = 1 + \frac{3}{2} v_s \frac{N_s^2 - 1}{N_s^2 + 2} \tag{57}$$

This is the well known refractive index for Rayleigh scattering (see, for example, van de Hulst [1957]). Using the following expression, due to Lorentz, for the polarizability of a sphere

$$\alpha = a^3 \frac{N_s^2 - 1}{N_s^2 + 2}$$

the refractive index is given by the more familiar equation

$$N_B = 1 + 2\pi \rho_o \alpha$$

It is immediately seen that if the spheres are non-absorbing, then so is the modified medium (since both N_s and N_B are real in this case).

The small sphere approximation for $\mu = \mu_s$ is equivalent to neglecting all orders of multipoles, except the first order electric dipole, in the Mie expansion. Thus the approximations are the same as those used in Rayleigh scattering theory and, consequently, the result is the same.

Another important special case is that of small, perfectly conducting spheres. It is important to note that this case is not covered by Rayleigh scattering. This is because as $N_s \to \infty$, the wavelength inside the sphere, $\frac{\lambda}{N_s}$, becomes infinitesimal. The condition for Rayleigh scattering, viz., that the radius be small compared to wavelength, is no longer satisfied inside the sphere. Therefore, we cannot treat this case by letting N_s

become infinitely large in equation (57). Instead, we see that the internal fields are zero for perfect conductors and, consequently, the Mie coefficients become

$$a_n^s = -\frac{j_n(\zeta)}{h_n(\zeta)}, \quad b_n^s = -\frac{[\zeta\, j_n(\zeta)]'}{[\zeta\, h_n(\zeta)]'}, \quad a_n^t = b_n^t = 0$$

For terms up to ζ^3 only, the Born approximation results now become

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x (1 - v_s)\left(1 - \frac{v_s}{8}\right) e^{iN_B kz}, \quad z > a$$

$$\langle \underline{E}(\underline{r}) \rangle = \hat{i}_x e^{ikz} - \hat{i}_x \frac{9}{8} v_s e^{-ikz}, \quad z < -a \tag{58}$$

and the refractive index is given by

$$N_B = 1 + \frac{3}{4} v_s \left[1 - \frac{v_s}{8}\right]^{-1}$$

For sparse concentrations of perfectly conducting spheres, we have

$$N_B \approx 1 + \frac{3}{4} v_s \tag{59}$$

In this case both the electric and the magnetic dipole terms of the Mie series are retained. Equation (59) shows that there is no attenuation at low frequencies in a medium containing small perfectly conducting spheres.

6.12 Sphere Size Comparable to Wavelength

When the radius of the sphere is comparable to the wavelength, the contribution of the higher order multipoles of the Mie series can no longer be neglected. The behavior of the medium should then be considered using the full solution given in Chapter 4 for the Born approximation. The refractive index will, in general, have an imaginary part also, indicating attenuation in the medium.

Table I shows the calculated values of the real and imaginary parts of the refractive index for perfectly conducting spheres in the range $\zeta = 0.1$ to 5.0.[4] These values have been calculated for three values of the fractional volume, v_s, occupied by the scatterers. This fractional occupied volume is a parameter indicative of the closeness of the packing, since

$$v_s = \frac{4}{3}\pi a^3 \rho_o = \frac{4}{3}\pi \left[\frac{a}{(1/\rho_o)^{1/3}}\right]^3$$

and $\left(\frac{1}{\rho_o}\right)^{1/3}$ is a measure of the average separation of the spheres. For large values v_s, the Born approximation is not valid and multiple scattering effects must be considered. These preliminary results are obtained using a desk calculator and we do not have enough information to plot a reliable graph for the propagation and attenuation constants. These constants are expected to show resonance effects characteristic of the Mie series coefficients.

6.2 Multiple Scattering Behavior

When the distribution of spheres in the medium is quite dense, multiple scattering effects can no longer be neglected and the treatment of Chapter 5

[4]Values of spherical Bessel and Hankel functions and trigonometric functions used in the computation are taken from Lowan, et. al. [1946] and Mathematical Tables [1959].

Table I

Real and imaginary parts of the refractive index in the Born approximation for perfectly conducting spheres

ζ	$v_s = 0.1$		$v_s = 0.01$		$v_s = 0.001$	
	$Re[N_B] - 1$	$Im[N_B]$	$Re[N_B] - 1$	$Im[N_B]$	$Re[N_B] - 1$	$Im[N_B]$
0.1	0.636	0.318×10^{-3}	6.57×10^{-3}	0.132×10^{-4}	6.65×10^{-4}	0.098×10^{-5}
0.2	0.069	0.986	7.70	0.442	7.88	1.02
0.3	0.078	3.47	8.25	3.362	8.26	3.34
0.5	0.096	38.0	9.30	16.32	9.26	16.0
1.0	0.041	3.56	4.06	1.43	4.05	1.23
2.0	0.018	40.58	2.20	41.3	2.20	40.9
5.0	0.976 - 1	14.80	0.29	26.2	0.30	15.7

should be used. The refractive index of the modified medium is governed by equations (41) and (43). For the case of small spheres, when both ζ and $N_s \zeta$ are very small, only the electric and magnetic dipole terms need be retained. The coefficients corresponding to these terms are

$$A^s_{\ell 1} = \frac{2i}{3} \frac{\mu}{\mu_\ell} \left[\frac{\mu_s - \mu_\ell}{\mu_s + 2\mu}\right] N_\ell \zeta^3$$

$$B^s_{\ell 1} = \frac{2i}{3} \left[\frac{\epsilon_s - \epsilon_\ell}{\epsilon_s + 2\epsilon}\right] \zeta^3$$

Using these values, equations (41) and (43) reduce to the following simpler form

$$3 \frac{\mu}{\mu_\ell} \left[\frac{\mu_s - \mu_\ell}{\mu_s + 2\mu}\right] N_\ell^2 + \left[\frac{\epsilon_s - \epsilon_\ell}{\epsilon_s + 2\epsilon}\right] (2 + N_\ell^2) = \frac{N_\ell^2 - 1}{v_s}$$

$$3 \frac{\epsilon}{\epsilon_\ell} \left[\frac{\epsilon_s - \epsilon_\ell}{\epsilon_s + 2\epsilon}\right] N_\ell^2 + \left[\frac{\mu_s - \mu_\ell}{\mu_s + 2\mu}\right] (2 + N_\ell^2) = \frac{N_\ell^2 - 1}{v_s} \qquad (60)$$

These equations give the refractive index, permeability, and dielectric constant of the modified medium.

For the case of perfectly conducting small spheres, equation (41) leads to the following equation when terms up to ζ^3 and $(N_\ell \zeta)^3$ are retained:

$$\left(\frac{7}{18} \zeta^2\right) N_\ell^4 - \left[\frac{1}{3}\left(1 + \frac{2}{v_s}\right) + \frac{23}{10} \zeta^2 + \frac{i}{9} \zeta^3\right] N_\ell^2 + \left[\frac{2}{3}\left(2 + \frac{1}{v_s}\right) + \frac{10}{3} \zeta^2 - \frac{i}{9} \zeta^3\right] = 0 \qquad (61)$$

At very low frequencies we have only one mode with the refractive index given by

$$N_{\ell}^2 = \frac{1 + 2v_s}{1 + v_s/2} ,$$

an expression derived earlier by Twersky [1962c]. For sparse concentrations this reduces to the result obtained from single scattering theory given by equation (59) as expected. The transmission and reflection coefficients reduce to

$$E^t = 1 - \frac{9}{8} v_s$$

$$E^r = - \frac{9}{8} v_s$$

and satisfy the boundary condition at $z = 0$ as expected. Since the transmission coefficient for normal incidence is given by (see Jordan [1960])

$$E^t = \frac{2\eta_{\ell}}{\eta_{\ell} + \eta} , \quad \eta_{\ell} = \left(\frac{\mu_{\ell}}{\epsilon_{\ell}}\right)^{\frac{1}{2}} , \quad \eta = \left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}$$

we can solve for μ_{ℓ} and ϵ_{ℓ} to the first power of v_s and get

$$\frac{\mu_{\ell}}{\mu} = 1 - \frac{3}{2} v_s$$

$$\frac{\epsilon_{\ell}}{\epsilon} = 1 + 3 v_s$$

These expressions agree with the results of Twersky [1962c]. For higher frequencies, the convergence is slower and more terms of equation (61) have to be considered. In this case more than one mode will be obtained due to the spatial dispersion effects.

7. Discussion

Wave propagation in a random medium is a very important problem and has, consequently, received considerable attention in the literature. A theoretical study of the problem requires a mathematical model that describes the properties of the random medium. Several such models are possible and the choice depends upon the type of problem being considered. One model is that in which some properties of the medium (such as the density, refractive index, etc.) are considered to be random functions of position. Another model, one that has been adopted in this investigation, considers the randomness as being due to the presence of distinct scattering objects which have electromagnetic properties different from those of the background medium. In this case the problem is formulated in terms of multiply scattered fields which satisfy a wave equation and boundary conditions on the surface of the scatterers. We have made such a formulation which is valid for scatterers of arbitrary size, shape and properties.

We have next specialized the treatment to the case of spherical scatterers. Previous work on multiple scattering by spheres is largely restricted to the case of small spheres and the fields interior to the sphere are ignored. The problem thus reduces to that of point scatterers. This introduces a singularity in the kernel of the integral equation and there is some ambiguity in treating such integrals since the results depend upon the shape of the volume excluded in the Cauchy principal value technique (Waterman and Truell [1961]). Such ambiguity no longer exists when we allow the scatterers to have finite size since the singularities are not in the volume of integration. This is a mathematically more satisfying approach. Our treatment is quite exact in that the full multipole solution of scattering by a sphere is used, taking into account the fields both inside and outside the spheres. Thus the treatment is valid for all frequencies. The results, when specialized to low frequencies, agree with the results obtained by other authors, including the well-known results of Rayleigh scattering theory.

We have considered propagation in weakly as well as strongly random media. Although the problem has been completely solved for weakly random media, the results obtained for strongly random media are only partially complete. Nevertheless, a dispersion relation has been obtained for the scattering medium. It is found that due to spatial dispersion, more than one mode can propagate in the medium. All the modes have a polarization identical to that of the incident wave in the case of normal incidence.

The present work has developed some of the basic techniques for dealing with multiple scattering problems, in particular, scattering by spheres. The vector nature of the problem is fully taken into account. This work can be extended in several directions. The full significance of the theory can be appreciated only when it is applied to get numerical results for specific cases. The complexity of the formulas necessitates the use of computers for solving the equations. This should provide an interesting and challenging problem for computer programming and it is hoped it will soon be taken up. The theoretical treatment can be extended in several directions. First, we have omitted from consideration a slab region near the boundary separating the scattering region. This region needs to be investigated with the hope of getting more information about the different modes. Second, the theory needs to be generalized to oblique incidence. It appears that the integrals involved in oblique incidence could be solved analytically along the lines of those treated here. Finally, the computation of power and energy appears to be a straightforward extension of this work.

Bibliography

1. Booker, H. G., "A Theory of Scattering by Nonisotropic Irregularities with Application to Radar Reflections from the Aurora," J. Atmos. Terr. Phys., V. 8, 204-221 (1956).

2. Born, M., and E. Wolf, Principles of Optics, Pergamon Press, New York (1959).

3. Chernov, L. A., Wave Propagation in a Random Medium, English Translation by R. A. Silverman, McGraw-Hill Book Co., Inc., New York (1961).

4. Cruzan, O. R., "Translation Addition Theorems for Spherical Vector Wave Functions," Quart. of App. Math., V. 20, 33-40 (1962).

5. Foldy, L. L., "The Multiple Scattering of Waves," Phys. Rev., V. 67, No. 3 and 4, 107-119 (1945).

6. Goodrich, R. F., B. A. Harrison, R. E. Kleinman, and T. B. A. Senior, "Studies in Radar Cross Sections XLVII -- Diffraction and Scattering by Regular Bodies -- 1: The Sphere," Department of Electrical Engineering, University of Michigan, Ann Arbor, Michigan (December 1961).

7. Gumprecht, R. O., and C. M. Sliepcevich, Tables of Riccati Bessel Functions for Large Arguments and Orders, Engineering Research Institute, University of Michigan, Ann Arbor, Michigan (1951).

8. Jordan, E. C., Electromagnetic Waves and Radiating Systems, Prentice-Hall, Englewood Cliffs, N. J. (1950).

9. Keller, J. B., "Wave Propagation in Random Media," Research Report No. EM-164, Institute of Mathematical Sciences, New York University, N. Y. (November 1960).

10. Lax, M., "Multiple Scattering of Waves," Rev. of Mod. Phys., V. 23, No. 4, 287-310 (1951).

11. Lax, M., "Multiple Scattering of Waves. II. The Effective Field in Dense Systems," Phys. Rev., V. 85, No. 4, 621-629 (1952).

12 Lowan, A. N., P. M. Morse, H. Feshbach, and M. Lax, Scattering and Radiation from Circular Cylinders and Spheres. Tables of Amplitudes and Phase Angles, U. S. Navy Department, Office of Research and Inventions (July 1946).

13. Magnus, W., and F. Oberhettinger, Functions of Mathematical Physics, Chelsea Publishing Company, New York (1949).

14. Mathematical Tables from Handbook of Chemistry and Physics, Chemical Rubber Publishing Co., Cleveland, Ohio (1959).

15. Mie, G., "Beiträge zur Optik trüber Medien," Ann. Physik., V. 25, 377-445 (1908).

16. Morse, P. M., and H. Feshbach, Methods of Theoretical Physics, Part I and II, McGraw-Hill Book Co., Inc., New York (1953).

17. Stratton, J. A., Electromagnetic Theory, McGraw-Hill Book Co., Inc., New York (1941).

18. Twersky, V., "On Multiple Scattering of Waves," J. Research NBS, V. 64D, 715-730 (1960).

19. Twersky, V., "Multiple Scattering of Waves and Optical Phenomena," J. Opt. Soc. Amer., V. 52, No. 2, 145-171 (February 1962).

20. Twersky, V., a. "On a General Class of Scattering Problems," b. "Scattering of Waves by Random Distributions. I. Free-Space Scatterer Formalism," c. "On Scattering of Waves by Random Distributions. II. Two-Space Scatterer Formalism," J. Mathematical Physics, V. 3, No. 4, 700-734 (July-August 1962).

21. van de Hulst, H. C., Light Scattering by Small Particles, John Wiley and Sons, Inc., New York (1957).

22. van der Pol, B., "A Generalization of Maxwell's Definition of Solid Harmonics to Waves in n-Dimensions," Physica, V. 3, No 6, 393-394 (1936).

23. Waterman, P. C., and R. Truell, "Multiple Scattering of Waves," J. Mathematical Physics, V. 2, No. 4, 512-537 (July-August 1961).

24. Yaglom, A. M., An Introduction to the Theory of Stationary Random Functions, English Translation by R. A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, N. J. (1962).

25. Yeh, K. C., "Propagation of Spherical Waves Through an Ionosphere Containing Anisotropic Irregularities," J. Research NBS, V. 66D, No. 5, 621-636 (1962).

Appendix I

We shall develop here a formula for converting a volume integral of the form

$$\int_V dv\; e^{ik'z}\, P_n(\cos\theta)\, h_n(kr) \tag{A1}$$

to a surface integral.

The functions $e^{ik'z}$ and $P_n(\cos\theta)\, h_n(kr)$ satisfy the Helmholtz wave equations

$$(\nabla^2 + k'^2)\, e^{ik'z} = 0$$

$$(\nabla^2 + k^2)\, P_n(\cos\theta)\, h_n(kr) = 0$$

everywhere except at the origin. If $k' \neq k$, we have

$$\int_V dv\; e^{ik'z}\, P_n(\cos\theta)\, h_n(kr) = \frac{1}{k'^2-k^2}\int_V [e^{ik'z}\,\nabla^2\left\{P_n(\cos\theta)\, h_n(kr)\right\} -$$

$$- P_n(\cos\theta)\, h_n(kr)\nabla^2\, e^{ik'z}]\, dv$$

$$= \frac{1}{k'^2-k^2}\int_\sigma [e^{ik'z}\,\nabla\left\{P_n(\cos\theta)\, h_n(kr)\right\} - P_n(\cos\theta)\, h_n(kr)\nabla e^{ik'z}]\cdot \underline{dS} \tag{A2}$$

where the volume V is enclosed in the surface σ and $\underline{dS}$ is in the direction of the outward normal. We have made use of Green's theorem here. Let us operate both sides of this equation by the operator

$$\lim_{\Delta\to 0} \frac{1}{\Delta}\left\{\int_{k-\Delta/2}^{k+\Delta/2} dk'\,[\quad]\right\}, \qquad \Delta > 0$$

Since this operator involves only k' over a finite domain, we can interchange the k'-integral with the spatial integral and also the limit with the spatial integral. This leads to the following equation

$$\int_V dv\ P_n(\cos\theta)\ h_n(kr)\ [\lim_{\Delta\to 0} \frac{1}{\Delta} \int_\Delta dk'\ e^{ik'z}]$$

$$= \int_\sigma [\left\{ \lim_{\Delta\to 0} \frac{1}{\Delta} \int_\Delta dk' \frac{e^{ik'z}}{k'^2-k^2} \right\} \nabla \left\{ P_n(\cos\theta)\ h_n(kr) \right\}$$

$$- P_n(\cos\theta)\ h_n(kr) \nabla \left\{ \lim_{\Delta\to 0} \frac{1}{\Delta} \int_\Delta dk' \frac{e^{ik'z}}{k'^2-k^2} \right\}] \cdot \underline{dS}$$

where integration over Δ implies the limits $(k - \Delta/2)$ to $(k + \Delta/2)$. We shall consider a general case here in which k, the propagation constant of an arbitrary medium (not necessarily free space), may be complex. The standard method of integration in the theory of complex variables leads us to the equation

$$\lim_{\Delta\to 0} \frac{1}{\Delta} \int_\Delta dk'\ e^{ik'z} = e^{ikz}$$

Thus the operator merely reduces the left hand side of equation (A2) to the form (A1) in which we are interested.

On the right hand side we have

$$\lim_{\Delta\to 0} \frac{1}{\Delta} \int_\Delta dk' \frac{e^{ik'z}}{k'^2-k^2} = \frac{1}{2k} \lim_{\Delta\to 0} \frac{1}{\Delta} \int_\Delta [\frac{e^{ik'z}}{k'-k} - \frac{e^{ik'z}}{k'+k}]\ dk'$$

$$= \frac{1}{2k} \lim_{\Delta\to 0} \frac{1}{\Delta} [e^{ikz} \int_\Delta \frac{e^{i(k'-k)z}}{k'-k}\ dk' - e^{-ikz} \int_\Delta \frac{e^{i(k'+k)}}{k'+k}\ dk']$$

The first term has a singularity in the path of integration. We, therefore, take the Cauchy principal value as follows

$$\lim_{\Delta\to 0} \frac{1}{\Delta} \int_{k-\Delta/2}^{k+\Delta/2} \frac{e^{i(k'-k)z}dk'}{k'-k} = \lim_{\Delta\to 0} \frac{1}{\Delta} \left[\lim_{\epsilon\to 0} \left\{ \int_{-\frac{\Delta}{2}}^{-\epsilon} \frac{e^{ik''z}dk''}{k''} + \int_{\epsilon}^{\Delta/2} \frac{e^{ik''z}dk''}{k''} \right\}\right]$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} \left[\lim_{\epsilon - 0} \left\{ \ln(\frac{\epsilon}{\Delta/2}) + iz(\frac{\Delta}{2} - \epsilon) + \frac{(iz)^2}{2.2!} (\epsilon^2 - \frac{\Delta^2}{4}) + \ldots \right.\right.$$

$$\left.\left. + \ln(\frac{\Delta/2}{\epsilon}) + iz(\frac{\Delta}{2} - \epsilon) + \frac{(iz)^2}{2.2!} (\frac{\Delta^2}{4} - \epsilon^2) + \ldots \right\}\right]$$

$$= i\,z$$

The second term has no singularity in the path of integration and we have

$$\lim_{\Delta\to 0} \frac{1}{\Delta} \int_{\Delta} \frac{e^{i(k'+k)z}}{k'+k} dk' = \lim_{\Delta\to 0} \frac{1}{\Delta} \left[\ln(k'+k) + iz(k'+k) + \frac{(iz)^2}{2.2!} (k'+k)^2 + \ldots \right]_{\Delta}$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} \left[\ln(\frac{1+\Delta/4k}{1-\Delta/4k}) + i2kz(2\,\Delta/4k) + \frac{(i2kz)^2}{2.2!} \left\{(1 + \Delta/4k)^2 - (1 - \Delta/4k)^2\right\} + \ldots \right]$$

$$= \frac{1}{2k} + \frac{i2kz}{2k} + \frac{1}{2k} \frac{(i2kz)^2}{2!} + \frac{1}{2k} \frac{(i2kz)^3}{3!} + \ldots$$

$$= \frac{1}{2k} e^{i2kz}$$

Putting all the terms together we get

$$\lim_{\Delta\to 0} \frac{1}{\Delta} \int_{\Delta} dv' \, \frac{e^{ik'z}}{k'^2 - k^2} = \frac{e^{ikz}}{2k} \left[iz - \frac{1}{2k} \right]$$

This gives us the desired formula for converting volume integral to surface integral as below

$$\int_V dv \, e^{ikz} \, P_n(\cos\theta) \, h_n(kr)$$

$$= \int_\sigma \Big[e^{ikz} \left(\frac{iz}{2k} - \frac{1}{4k^2}\right) \nabla \left\{P_n(\cos\theta) \, h_n(kr)\right\}$$

$$- P_n(\cos\theta) \, h_n(kr) \nabla \left\{ e^{ikz} \left(\frac{iz}{2k} - \frac{1}{4k^2} \right) \right\} \Big] \cdot \underline{dS} \qquad \text{(A3)}$$

Appendix II

We have made use of the relation

$$P_n(\cos\theta)\, h_n(kr) = (-i)^n\, P_n\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) h_o(kr) \qquad \text{(A4)}$$

to carry out some of the integrations involved in this investigation. We give below a proof by induction of this relation based on the work of Balth van der Pol [1936].

The relation is easily seen to hold for n = 0 and 1 since

$$P_o(\cos\theta) = 1 \,,\; h_o(kr) = \frac{e^{ikr}}{ikr}$$

$$P_1(\cos\theta) = \cos\theta \,,\; h_1(kr) = \frac{e^{ikr}}{(ikr)^2}(kr + i)$$

Let us assume that the relation is true for n and n-1. We shall show that it is true for n+1, i.e., that

$$P_{n+1}(\cos\theta)\, h_{n+1}(kr) = (-i)^{n+1}\, P_{n+1}\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) h_o(kr)$$

From the recurrence relations (see Morse and Feshbach [1953])

$$\frac{\partial}{\partial (kr)} h_n(kr) = h_{n-1}(kr) - \frac{n+1}{kr} h_n(kr)$$

$$\sin\theta \frac{\partial}{\partial\theta} P_n(\cos\theta) = \frac{n(n+1)}{2n+1}\left[P_{n+1}(\cos\theta) - P_{n-1}(\cos\theta)\right]$$

and

$$z = r\cos\theta \,,\; \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

we derive the following relation

$$\frac{\partial}{\partial(kz)} P_n(\cos\Theta)\, h_n(kr) = \frac{1}{2n+1}\left[nP_{n-1}(\cos\Theta)h_{n-1}(kr) - (n+1)P_{n+1}(\cos\Theta)h_{n+1}(kr)\right]$$

$$\therefore P_{n+1}(\cos\Theta)h_{n+1}(kr) = \frac{n}{n+1} P_{n-1}(\cos\Theta)h_{n-1}(kr) - \frac{2n+1}{n+1}\frac{\partial}{\partial(kz)} P_n(\cos\Theta)h_n(kr)$$

Since we have assumed the relation (A4) to be true for n and (n-1), this reduces to

$$P_{n+1}(\cos\Theta)\, h_{n+1}(kr) = \frac{n}{n+1}\left[(-i)^{n-1} P_{n-1}\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) h_o(kr)\right] -$$

$$- \frac{2n+1}{n+1}\frac{\partial}{\partial(kz)}\left[(-i)^n P_n\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) h_o(kr)\right]$$

$$= (-i)^{n-1}\left[\frac{n}{n+1} P_{n-1}\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) - \frac{2n+1}{n+1}\frac{1}{ik}\frac{\partial}{\partial z} P_n\left(\frac{1}{ik}\frac{\partial}{\partial z}\right)\right] h_o(kr)$$

Now we use the recurrence relation

$$\frac{2n+1}{n+1}\, x\, P_n(x) - \frac{n}{n+1} P_{n-1}(x) = P_{n+1}(x) ,$$

with $x = \frac{1}{ik}\frac{\partial}{\partial z}$. This immediately reduces the above equation to

$$P_{n+1}(\cos\Theta)\, h_{n+1}(kr) = (-i)^{n+1} P_{n+1}\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) h_o(kr) .$$

Thus we see that (A4) is true for (n+1) if it is true for n and (n-1). But we have already seen that it is true for n = 0 and 1. The proof by induction is, therefore, complete.

Appendix III

We consider here the "two-exterior" problem of scattering by a sphere. Let an incident field $\hat{i}_x e^{ik_\ell z}$ excite a sphere of radius $\underline{a}$ and electromagnetic properties specified by k_s, μ_s and ϵ_s. The scattered field travels in the medium specified by k, μ and ϵ. Following Stratton [1943], we write the incident, scattered and transmitted fields in terms of spherical vector waves as follows:

$$\underline{E}^i(\underline{r}) = \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left[\underline{m}^1_{oln}(\underline{r},k_\ell) - i\, \underline{n}^1_{eln}(\underline{r},k_\ell) \right]$$

$$\underline{E}^s(\underline{r}) = \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left[A^s_{\ell n}\, \underline{m}^3_{oln}(\underline{r},k) - i\, B^s_{\ell n}\, \underline{n}^3_{eln}(\underline{r},k) \right]$$

$$\underline{E}^t(\underline{r}) = \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left[A^t_{\ell n}\, \underline{m}^1_{oln}(\underline{r},k_s) - i\, B^t_{\ell n}\, \underline{n}^1_{eln}(\underline{r},k_s) \right]$$

The boundary conditions require continuity of electric and magnetic fields on the surface of the sphere. These can be written as

$$[\hat{i}_r \times (\underline{E}^i + \underline{E}^s - \underline{E}^t)]_{r=a} = 0 \tag{A5}$$

$$[\hat{i}_r \times (\underline{H}^i + \underline{H}^s - \underline{H}^t)]_{r=a} = 0 \tag{A6}$$

The vector wave functions can be written in terms of spherical coordinates as follows

$$\underline{m}^{1,3}_{oln}(\underline{r},k) = \hat{i}_\theta\left[\frac{1}{\sin\theta}\, z_n(kr)\, P^1_n(\cos\theta)\cos\phi\right] - \hat{i}_\phi\left[z_n(kr)\,\frac{\partial P^1_n}{\partial\theta}\sin\phi\right]$$

$$\underline{n}_{e1n}^{1,3}(\underline{r},k) = \hat{i}_r\left[\frac{n(n+1)}{kr} z_n(kr) P_n^1(\cos\Theta)\cos\phi\right]$$

$$+ \hat{i}_\Theta\left[\frac{1}{kr}\frac{\partial}{\partial r}\{r\, z_n(kr)\}\frac{\partial P_n^1(\cos\Theta)}{\partial\Theta}\cos\phi\right]$$

$$- \hat{i}_\phi\left[\frac{1}{kr\sin\Theta}\frac{\partial}{\partial r}\{r\, z_n(kr)\} P_n^1(\cos\Theta)\sin\phi\right]$$

Here $z_n = j_n$ if the superscript is 1 and $z_n = h_n$ if the superscript is 3. Equation (A5) leads to two equations when the above expressions are substituted in it and the orthogonality of Legendre Polynomials is used. These are

$$j_n(k_\ell a) + A_{\ell n}^s h_n(ka) - A_{\ell n}^t j_n(k_s a) = 0 \tag{A7}$$

$$\frac{1}{k_\ell}[k_\ell a j_n(k_\ell a)]' + \frac{B_{\ell n}^s}{k}[ka\, h_n(ka)]' - \frac{B_{\ell n}^t}{k_s}[k_s a\, j_n(k_s a)]' = 0 \tag{A8}$$

The magnetic field is derived using the following standard relations:

$$\underline{H} = \frac{1}{i\omega\mu}\nabla\times\underline{E}\ ;\ \nabla\times\underline{m}(\underline{r},k) = k\,\underline{n}(\underline{r},k)\ ;\ \nabla\times\underline{n}(\underline{r},k) = k\,\underline{m}(\underline{r},k)$$

and the expansions of vector spherical waves

$$\underline{m}_{e1n}^{1,3}(\underline{r},k) = \hat{i}_\Theta\left[\frac{-1}{\sin\Theta}P_n(\cos\Theta)\, z_n(kr)\right] - \hat{i}_\phi\left[\frac{\partial P_n^1(\cos\Theta)}{\partial\Theta}\cos\phi\, z_n(kr)\right]$$

$$\underline{n}_{o1n}^{1,3}(\underline{r},k) = \hat{i}_r\left[\frac{n(n+1)}{kr} z_n(kr) P_n^1(\cos\Theta)\sin\phi\right]$$

$$+ \hat{i}_\Theta\left[\frac{1}{kr}\frac{\partial}{\partial r}\{r\, z_n(kr)\}\frac{\partial P_n^1(\cos\Theta)}{\partial\Theta}\sin\phi\right]$$

$$+ \hat{i}_\phi\left[\frac{1}{kr\sin\Theta}\frac{\partial}{\partial r}\{r\, z_n(kr)\} P_n^1(\cos\Theta)\cos\phi\right]$$

These expressions, together with equation (A6), lead to

$$\frac{k_\ell}{\mu_\ell}\, j_n(k_\ell a) + B^s_{\ell n}\, \frac{k}{\mu}\, h_n(ka) - B^t_{\ell n}\, \frac{k_s}{\mu_s}\, j_n(k_s a) = 0 \qquad \text{(A9)}$$

$$\frac{1}{\mu_\ell}\,[k_\ell a\; j_n(k_\ell a)]' + \frac{A^s_{\ell n}}{\mu}\,[ka\; h_n(ka)]' - \frac{A^t_{\ell n}}{\mu_s}\,[k_s a\; j_n(k_s a)]' = 0 \qquad \text{(A10)}$$

Solving equations (A7) - (A10) we get

$$A^s_{\ell n} = \frac{\mu}{\mu_\ell}\; \frac{\mu_s\, j_n(N_s\zeta)[N_\ell\zeta\, j_n(N_\ell\zeta)]' - \mu_\ell\, j_n(N_\ell\zeta)[N_s\zeta\, j_n(N_s\zeta)]'}{\mu\, h_n(\zeta)[N_s\zeta\, j_n(N_s\zeta)]' - \mu_s\, j_n(N_s\zeta)[\zeta\, h_n(\zeta)]'} \qquad \text{(A11)}$$

$$B^s_{\ell n} = \frac{\mu}{\mu_\ell N_\ell}\; \frac{\mu_s N_\ell^2\, j_n(N_\ell\zeta)[N_s\zeta\, j_n(N_s\zeta)]' - \mu_\ell\, N_s^2\, j_n(N_s\zeta)[N_\ell\zeta\, j_n(N_\ell\zeta)]'}{\mu\, N_s^2\, j_n(N_s\zeta)[\zeta\, h_n(\zeta)]' - \mu_s\, h_n(\zeta)[N_s\,\zeta\, j_n(N_s\zeta)]'} \qquad \text{(A12)}$$

$$A^t_{\ell n} = \frac{\mu_s}{\mu_\ell}\; \frac{\mu h_n(\zeta)[N_\ell\zeta\, j_n(N_\ell\zeta)]' - \mu_\ell\, j_n(N_\ell\zeta)[\zeta\, h_n(\zeta)]}{\mu h_n(\zeta)[N_s\,\zeta j_n(N_s\zeta)] - \mu_s\, j_n(N_s\zeta)[\zeta\, h_n(\zeta)]'} \qquad \text{(A13)}$$

$$B^t_{\ell n} = \frac{\mu_s N_s}{\mu_\ell N_\ell}\; \frac{\mu N_\ell^2\, j_n(N_\ell\zeta)[\zeta\, h_n(\zeta)]' - \mu_\ell\, h_n(\zeta)[N_\ell\,\zeta\, j_n(N_\ell\zeta)]'}{\mu N_s^2\, j_n(N_s\zeta)[\zeta\, h_n(\zeta)]' - \mu_s\, h_n(\zeta)[N_s\zeta\, j_n(N_s\zeta)]'} \qquad \text{(A14)}$$

The notation used is defined by

$$ka = \zeta\,, \quad k_\ell = N_\ell k\,, \quad k_s = N_s k$$

Now if we start with an incident field

$$\underline{H}^i(\underline{r}) = \hat{i}_x\, e^{ik_\ell z}$$

we can carry on the analysis exactly as above except that we now have the relation:

$$\underline{E}(\underline{r}) = \frac{i}{\omega\epsilon} \nabla \times \underline{H}(\underline{r})$$

It is easily seen that if the expansion coefficients are $C^s_{\ell n}$, $D^s_{\ell n}$, $C^t_{\ell n}$, $D^t_{\ell n}$ in this case, then they will be given by equations similar to (A11)-(A14) except that the μ's are replaced by ϵ's. Thus we can write these coefficients as follows:

$$C^s_{\ell n} = A^s_{\ell n} \; (\mu \rightarrow \epsilon) \; ; \qquad C^t_{\ell n} = A^t_{\ell n} \; (\mu \rightarrow \epsilon)$$

$$D^s_{\ell n} = B^s_{\ell n} \; (\mu \rightarrow \epsilon) \; ; \qquad D^t_{\ell n} = B^t_{\ell n} \; (\mu \rightarrow \epsilon) \tag{A15}$$

However, since

$$k^2 = \omega^2 \mu \epsilon \; , \qquad k_s^2 = \omega^2 \mu_s \epsilon_s \; , \qquad k_\ell^2 = \omega^2 \mu_\ell \epsilon_\ell$$

we see that

$$N_s^2 = \frac{\mu_s \epsilon_s}{\mu \epsilon} \; , \qquad N_\ell^2 = \frac{\mu_\ell \epsilon_\ell}{\mu \epsilon}$$

Using these relations, the following relationship between coefficients is easily established:

$$C^s_{\ell n} = N_\ell \frac{\mu_\ell}{\mu} B^s_{\ell n}$$

$$D^s_{\ell n} = N_\ell \frac{\mu_\ell}{\mu} A^s_{\ell n}$$

$$C^t_{\ell n} = \frac{\mu_\ell N_s}{\mu_s N_\ell} B^t_{\ell n} \tag{A16}$$

$$D^t_{\ell n} = \frac{\mu_\ell N_s}{\mu_s N_\ell} A^t_{\ell n}$$